EPSILON FACTORS FOR MEROMORPHIC CONNECTIONS AND GAUSS SUMS

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Abstract. Let E is be vector bundle with meromorphic connection on $\mathbb{P}^1/k$ for some field $k \subset \mathbb{C}$, and let $\mathbf{E}$ be the sheaf of horizontal sections on the analytic points of X. The irregular Riemann-Hilbert correspondence states that there is a canonical isomorphism between the De Rham cohomology of L and the 'moderate growth' cohomology of $\mathbf{L}$. Recent work of Beilinson, Bloch, and Esnault has shown that the determinant of this map factors into a product of local 'ε-factors' which closely resemble the classical ε-factors of Galois representations. In this paper, we show that ε-factors for rank one connections may be calculated explicitly by a Gauss sum. This formula suggests a deeper relationship between the De Rham ε-factor and its Galois counterpart.

1. Introduction

1.1. **Motivation.** The theory of ε-factors originates from a curious asymmetry in the functional equation of a Zeta function, first observed by Tate. Suppose that F is a local field and χ is a character of $F^\times$. If Φ is a test function on F, we may define a zeta function $Z(\Phi, \chi, s)$ in the usual way. The ratio $\Xi(\Phi, \chi, s) = \frac{Z(\Phi,\chi,s)}{L(\chi,s)}$, where $L(\chi, s)$ is the L function associated to χ, satisfies a functional equation:

$$\Xi(\widehat{\Phi}, \chi^\vee, 1-s) = \varepsilon(\chi, s, \psi)\, \Xi(\Phi, \chi, s).$$

Above, $\chi^\vee$ is the contragredient of χ and $\widehat{\Phi}$ is the Fourier transform of Φ; notice that the construction of the Fourier transform requires a choice of an additive character, which we denote by ψ. In particular, the functional equations of L and Z differ by an error term $\varepsilon(\chi, s, \psi)$ that only depends on χ and ψ.

We recall a few standard properties of $\varepsilon(\chi, s, \psi)$, taken from [10] chapter 6. Let U^j be the j^{th} unit subgroup of $F^\times$. Suppose that ψ has conductor $c(\psi)$, and let $a(\chi)$ be the least integer such that $\chi|_{U^a}$ is the trivial character. First of all, if the residue field of F has order q, then

$$\varepsilon(\chi, s, \psi) = q^{(\frac{1}{2}-s)(a(\chi)+c(\psi))}\varepsilon(\chi, 1/2, \psi).$$

In particular, $\varepsilon(\chi, s, \psi)$ is a monomial in the ring $\mathbb{C}[q^{s/2}, q^{-s/2}]$ (resp. $\overline{\mathbb{Q}}_\ell[q^{s/2}, q^{-s/2}]$, depending on the field of coefficients). For historical reasons, we will take the convention $\varepsilon(\chi, \psi) = q^{\frac{1}{2}c(\psi)}\varepsilon(\chi, 0, \psi)$; by our previous observation, this quantity uniquely determines $\varepsilon(\chi, s, \psi)$.

When χ is ramified, then $\varepsilon(\chi, \psi)$ may be calculated in terms of a Gauss sum. In particular, if dz is the Haar measure on F normalized so that the ring of integers $\mathfrak{o}$ has measure 1,

$$\varepsilon(\chi, \psi) = \int_{\gamma^{-1}\mathfrak{o}} \chi(z)^{-1}\psi(z)dz.$$

Above, γ is an element of $F^\times$ chosen to have valuation $a(\chi) + c(\psi)$.

1.2. **Geometric ε-factors.** If we consider the Tate ε-factor through the lens of class field theory, a very different picture emerges. Fix a separable algebraic closure $\bar{F}$ of F and let $\mathscr{W}_F \subset \mathrm{Gal}(\bar{F}/F)$ be the Weil group of F. Then, the abelianization of $\mathscr{W}_F$ is isomorphic to $F^\times$; in particular, we may think of the ε factor as an invariant of abelian Galois representations of $\mathscr{W}_F$.

A deep theory, due to Langlands and Deligne ([11]), shows that there exists a natural generalization of the ε factor to n dimensional semisimple smooth representations of $\mathscr{W}_F$. The Deligne-Langlands ε-factor, or 'local constant,' has been studied extensively as one of the core invariants of the Langlands corresponence. In particular, if we specialize to the case where F is a function field, then the Deligne-Langlands ε-factor has some very nice properties as a purely geometric invariant.

Suppose that X/k is an algebraic curve in positive characteristic with function field K, and let $\mathscr{F}$ be an irreducible smooth ℓ-adic sheaf of rank n on some dense open subset $j : V \hookrightarrow X$. Let $\bar{k}$ be an algebraic closure of the field of coefficients. We define

$$\varepsilon(X, j_*\mathscr{F}) = \det(R\Gamma_c(V \times_k \mathrm{Spec}(\bar{k}), j_*\mathscr{F}))^{-1}.$$

This is a graded line in degree $-\chi(X, \mathscr{F})$, where χ is the usual Euler characteristic. If $k = \mathbb{F}_q$, the geometric Frobenius, Frob_q, acts on $\varepsilon(X, \mathscr{F})$; in particular, the determinant of $-\mathrm{Frob}_q$ is a constant in $\bar{\mathbb{Q}}_\ell$.[1]

We can recover the local constant by restricting $\mathscr{F}$ to the Henselization $X_{(x)}$ of X at a closed point $x \in |X|$. If $X_{(x)} = \mathrm{Spec}(R)$, then any one form ν defines an additive character on R by $\psi_\nu(r) = \mathrm{Res}(r\nu)$. The local constants, denoted by $\varepsilon(X_{(x)}; (j_*\mathscr{F})|_{X_{(x)}}, \nu)$, are graded lines with coefficients in $\bar{\mathbb{Q}}_\ell$ along with a canonical action of Frob_q. For instance, suppose that F is the field of fractions of R and η_x is the generic point. Then, when $\mathscr{F}_{\eta_x}$ has rank 1, $\mathscr{F}_{\eta_x}$ defines an abelian representation χ of $\mathscr{W}_F$. In this case, ([17], théorème 3.1.5.4, (v)),

$$\varepsilon(\chi, \psi_\nu) = \mathrm{Tr}\left(-\mathrm{Frob}_q, \varepsilon(X_{(x)}, (j_*\mathscr{F})_{\eta_x}, \nu)\right).$$

The essential property of the geometric ε factor, conjectured by Deligne and proved by Laumon, is the product formula ([17], théorème 3.2.1.1):

$$\det(-\mathrm{Frob}_q, \varepsilon(X; j_*\mathscr{F})) = q^{C(1-g)r(\mathscr{F})} \prod_{x \in |X|} (\mathrm{Tr}(-\mathrm{Frob}_q, \varepsilon(X_{(x)}; (j_*\mathscr{F})|_{X_{(x)}}, \nu|_{X_{(x)}})) \tag{1.2.1}$$

Above, ν is a global meromorphic 1 form, C is the number of connected components of $X \times_k \bar{k}$, $r(\mathscr{F})$ is the generic rank of $\mathscr{F}$, and g is the genus of X.

1.3. **De Rham ε-Factors.** Given the geometric nature of the Deligne-Langlands ε-factor, one might ask whether there is an analogous invariant for De Rham cohomology. This question dates back to Laumon ([17]), in which he cites Witten's proof of the Morse inequalities as the motivation for his proof of the product formula (1.2.1). Early work on the subject was presented by Deligne at an IHES seminar in 1984; however, interest in De Rham ε-factors has been revived by the recent work of Beilinson, Bloch and Esnault ([2], [3]).

[1] In [17], $\varepsilon(X, \mathscr{F})$ is defined as the determinant of $-\mathrm{Frob}_q$. Here, it is preferable to think of $\varepsilon(X, \mathscr{F})$ as a line with a canonical Frobenius endomorphism.

In order to get a handle on the De Rham theory, it is best to work globally to locally. In particular, suppose that X/k is a smooth projective curve with coefficients in $k \subset \mathbb{C}$, and E is a meromorphic connection that is smooth on $j : V \hookrightarrow X$. Let $\mathbf{E}_{\mathbb{C}}$ be the perverse sheaf of horizontal sections of E^{an} on V^{an}. According to the irregular Riemann-Hilbert correspondence (see section 2.4), there is a Stokes filtration $\{\mathbf{E}^*\}$ on sectors around the singular points of E. If we let $\mathbf{E}^0$ be the zero filtered part, there is a natural isomorphism[2]

$$H^*_{\mathrm{DR}}(X;E) \otimes_k \mathbb{C} \cong H^*(X^{an}; j_*\mathbf{E}^0) \tag{1.3.1}$$

(theorem 2.8).

The situation becomes more interesting if there is a reduction of structure for $\mathbf{E}_{\mathbb{C}}$ to a field $M \subset \mathbb{C}$, denoted $\mathbf{E}_M$, that is compatible with the stokes filtration at infinity. We define the global ε factor of the pair $(E, \mathbf{E}_M)$ to be the pair of lines $(\varepsilon_{\mathrm{DR}}(X; j_*E), \varepsilon_B(X^{an}; j_*(\mathbf{E}^*_M)))$ defined by

$$\begin{aligned} \varepsilon_{\mathrm{DR}}(X; j_*E) &= \det(H^*_{\mathrm{DR}}(X; j_*E)) \\ \varepsilon_B(X^{an}; j_*(\mathbf{E}^*_M)) &= \det(H^*(X^{an}, j_*\mathbf{E}^0_M)). \end{aligned} \tag{1.3.2}$$

Notice that $\varepsilon_{\mathrm{DR}}(X; j_*E)$ and $\varepsilon_B(X^{an}; j_*\mathbf{E}^*_M)$ are graded lines in degree $-\chi(X;E)$. Furthermore, there is a canonical isomorphism

$$\varepsilon_{\mathrm{DR}}(X;E) \otimes_k \mathbb{C} \cong \varepsilon_B(X^{an}; j_*\mathbf{E}^*_M) \otimes_M \mathbb{C} \tag{1.3.3}$$

defined by (1.3.1). If we choose k and M to be sufficiently small, this map contains non-trivial arithmetic data.

In [2] and [3], Beilinson, Bloch and Esnault demonstrate that there are local ε-factorizations[3] of $\varepsilon_B(X^{an}; j_*\mathbf{E}^*_M)$ and $\varepsilon_{\mathrm{DR}}(X; j_*E)$. Specifically, for each closed point $x \in X(k)$, we may localize j_*E to a formal connection $\hat{E}_x$ on the completion $X_{(x)}$ of X at x and $j_*\mathbf{E}^*_M$ to a filtered local system $(\mathbf{E}^*_M)_x$ on a small analytic neighborhood Δ_x containing x. Then, there is a theory of local ε factors for $X_{(x)}$ (resp. Δ_x): if we fix a global one form ν, there are canonical isomorphisms

$$\begin{aligned} \varepsilon_{\mathrm{DR}}(X; j_*E) &\cong \bigotimes_{x \in X} \varepsilon(X_{(x)}; \hat{E}_x, \nu) \\ \varepsilon_B(X^{an}; j_*\mathbf{E}^*_M) &\cong \bigotimes_{x \in \mathbb{P}^1} \varepsilon(\Delta_x; (\mathbf{E}^*_M)_x, \nu). \end{aligned} \tag{1.3.4}$$

Using a local Fourier transform, Bloch and Esnault have shown that there exist isomorphisms

$$\varepsilon(X_{(x)}; \hat{E}_x, \nu) \otimes_k \mathbb{C} \cong \varepsilon(\Delta_x; (\mathbf{E}^*_M)_x, \nu) \otimes_M \mathbb{C}$$

that are compatible with (1.3.3). Thus, we have a theory of local Betti and De Rham ε-factors that resemble the Deligne-Langlands local constants. The main cosmetic difference is that now we have two lines that are identified by a canonical Riemann-Hilbert isomorphism, whereas before we had a single line with a canonical Frobenius endomorphism. In the process, however, the connection between ε-factors and representation theory becomes obscured.

In this paper, we will show that the De Rham and Betti ε-factors for rank one connections may be calculated in terms of a Gauss sum. This problem was originally suggested by Bloch and Esnault in [5], and our approach owes much to their study

[2] Here, we take the standard perverse shift for De Rham and Betti cohomology

[3] see definition 3.4 for a formal definition

of Gauss-Manin determinants. Furthermore, the work of Bushnell *et. al.* ([8], [9]) suggests that there should be a non-commutative Gauss sum formula for ε-factors in higher rank; using the 'induction' axiom for ε-factors and the Gauss sum formula in rank 1, we have seen positive results in this direction ([7]).

1.4. **Results.** In rank one, the Stokes filtration is entirely determined by a Morse function. Therefore, most of our data will consist of pairs $(\mathscr{L}, \boldsymbol{\mathcal{L}})$, called 'Betti structures': $\mathscr{L}$ is a holonomic $\mathcal{D}$-module and $\boldsymbol{\mathcal{L}}$ is a perverse sheaf with coefficients in M. These objects are related by a fixed isomorphism $\mathrm{DR}(\mathscr{L}) \cong \boldsymbol{\mathcal{L}} \otimes_M \mathbb{C}$. The Morse function α is a meromorphic function determined by $\mathscr{L}$. In the global case, we denote the 'rapid decay' complex of $(\boldsymbol{\mathcal{L}}, \alpha)$ on V^{an} by $R\Gamma_c(V^{an}; \boldsymbol{\mathcal{L}}, \alpha)$. In particular, there is a natural quasi-isomorphism

$$R\Gamma_c(V^{an}; \boldsymbol{\mathcal{L}}, \alpha) \otimes_M \mathbb{C} \cong R\Gamma_c(V; \mathscr{L}) \otimes_k \mathbb{C}.$$

It is instructive to consider the case where the underlying scheme is $\mathrm{Spec}(k)$. Then, $\mathscr{L}$ is a k line, $\boldsymbol{\mathcal{L}}$ is an M line, and there is a fixed map $\mathscr{L} \otimes_k \mathbb{C} \cong \boldsymbol{\mathcal{L}} \otimes_M \mathbb{C}$. It is easy to see that isomorphism classes of pairs $(\mathscr{L}, \boldsymbol{\mathcal{L}})$ in degree 0 correspond to double cosets in $k^\times \backslash \mathbb{C}^\times / M^\times$; if $\xi \in \mathbb{C}$ is a coset representative, we denote the corresponding pair of lines in degree d by $(\xi)[-d]$. There is a natural tensor structure on such pairs, which amounts to $(\xi)[-d] \otimes (\xi')[-d'] = (\xi\xi')[-d-d']$.

Returning to the case where V is a quasi-projective variety, if $\mathscr{L}$ is non-singular at a point $i_x : x \to V$, we denote

$$(\mathscr{L}, \boldsymbol{\mathcal{L}}; x) = (i_x^* \mathscr{L}[-\dim(V)], i_x^* \boldsymbol{\mathcal{L}}[-\dim(V)]).$$

Notice that the shift is necessary to ensure that the lines are in degree 0.

First, we consider the local picture. The pair $(\hat{L}, \hat{\mathbf{L}})$ consists of the following: $\hat{L}$ is a formal holonomic $\mathcal{D}$-module with coefficients in $\mathfrak{o} = k[[t]]$; and $\hat{\mathbf{L}}$ is a perverse sheaf on an analytic disc Δ that is constructible with respect to the stratification $\{0\} \subset \Delta$. If j is the inclusion of the generic point of $\mathrm{Spec}(\mathfrak{o})$, it will suffice to consider the case where $\hat{L} = j_! j^* \hat{L}$. We suppose that $\hat{L}$ has irregularity index f at 0.

We use local class field theory to define a 'character sheaf' $(\mathscr{L}, \boldsymbol{\mathcal{L}})$ associated to $(\hat{L}, \hat{\mathbf{L}})$. As before, $\{U^i \subset \mathfrak{o}^\times : i \geq 0\}$ are the congruence subgroups with $U^0(k) \cong \mathfrak{o}^\times$; furthermore, we let $F^\times$ be the group ind-scheme defined by $\coprod_{n \in \mathbb{Z}} t^n U^0$. Then, $(\mathscr{L}, \boldsymbol{\mathcal{L}})$ is an invariant Betti structure on $F^\times / U^{f+1}$: $\mathscr{L}$ is an invariant line bundle with connection, and $\boldsymbol{\mathcal{L}}$ is an invariant local system with equivariant Morse function β. As in the arithmetic theory, any non-zero one form $\nu \in \Omega^1_{\mathfrak{o}/k}$ determines a Fourier sheaf $(\mathscr{F}_\nu, \boldsymbol{\mathcal{F}}_\nu, \psi_\nu)$. Let $c(\nu)$ be the order of the zero (or pole) of ν, and define $a(\mathscr{L}) = f + 1$.[4] We define a Gauss sum by

$$\begin{aligned}
\tau(\mathscr{L}, \boldsymbol{\mathcal{L}}; \nu) &= (\tau_{\mathrm{DR}}(\mathscr{L}; \nu), \tau_B(\boldsymbol{\mathcal{L}}, \beta; \nu)) \\
\tau_{\mathrm{DR}}(\mathscr{L}; \nu) &= R\Gamma_c(\gamma^{-1} U / U^{f+1}; \mathscr{L} \otimes_{\mathcal{O}} \mathscr{F}_\nu) \\
\tau_{\mathrm{DR}}(\boldsymbol{\mathcal{L}}, \beta; \nu) &= R\Gamma_c((\gamma^{-1} U / U^{f+1})^{an}; \boldsymbol{\mathcal{L}} \otimes_M \boldsymbol{\mathcal{F}}_\nu, \beta + \psi_\nu).
\end{aligned}$$

Above, γ is an element of $F^\times$ of degree $c(\nu) + a(\mathscr{L})$.

Theorem 1.1. *$\tau(\mathscr{L}, \boldsymbol{\mathcal{L}}; \nu)$ is a pair of lines in degree 0. Furthermore, there exists $\delta \in k$ and $g \in F^\times$ of degree $-c(\nu) - a(\mathscr{L})$ determined by $\mathscr{L}$ with the following property:*

[4]In particular, the trivial connection has $a = 1$.

(1) *when* $a = 1$,

$$\tau(\mathscr{L}, \boldsymbol{\mathcal{L}}; \nu) = (\Gamma(\delta))^{-1} \otimes (\mathscr{L}, \boldsymbol{\mathcal{L}}, g),$$

where Γ *is the usual gamma function;*

(2) *when* $a = a(\mathscr{L}) > 1$,

$$\tau(\mathscr{L}, \boldsymbol{\mathcal{L}}; \nu) = (e^{-\mathrm{Res}(g\nu)}) \otimes (\sqrt{\frac{\delta}{2\pi}})^a \otimes (\sqrt{-1})^{\lfloor \frac{a}{2} \rfloor} \otimes (\mathscr{L}, \boldsymbol{\mathcal{L}}, g).$$

This theorem is proved in section 5, theorem 5.4. We define

$$\varepsilon(\mathscr{L}, \boldsymbol{\mathcal{L}}; \nu) = (2\pi\sqrt{-1})^{c(\nu)} \otimes \tau(\mathscr{L}^\vee, \boldsymbol{\mathcal{L}}^\vee; \nu)[-c(\nu) - a(\mathscr{L})].$$

Our main theorem is that the local ε-factors satisfy a global product formula on $\mathbb{P}^1$. We now suppose that L is a line bundle with connection on $V \subset X = \mathbb{P}^1$, and $(L, \mathbf{L})$ is a Betti structure for L. Denote the inclusion of V in X by j. If $x \in X(k)$, we denote the localization of $(j_! L, j_! \mathbf{L})$ to $\mathrm{Spec}(\hat{\mathcal{O}}_{X,x})$ (resp. Δ_x) by $(\hat{L}_x, \hat{\mathbf{L}}_x)$. Let $(\mathscr{L}_x, \boldsymbol{\mathcal{L}}_x)$ be the character sheaf associated to $(\hat{L}_x, \hat{\mathbf{L}}_x)$. Define

$$\varepsilon(x; \hat{L}_x, \hat{\mathbf{L}}_x; \nu) = \begin{cases} (L, \mathbf{L}; x)^{c(\nu)}, & x \in V \\ \varepsilon(\mathscr{L}_x^\vee, \boldsymbol{\mathcal{L}}_x^\vee; \nu), & x \in X \backslash V, \end{cases}$$

where $(\mathscr{L}^\vee, \boldsymbol{\mathcal{L}}^\vee)$ is the pullback of $(\mathscr{L}, \boldsymbol{\mathcal{L}})$ by the inverse map on $F^\times$.

Theorem 1.2. *There is a canonical isomorphism of graded lines*

$$\varepsilon(X; j_! L, j_! \mathbf{L}) \cong (2\pi\sqrt{-1}) \bigotimes_{x \in \mathbb{P}^1} \varepsilon(\hat{L}_x, \hat{\mathbf{L}}_x; \nu).$$

Notice that this theorem, proved in section 6.3, specializes to the index formula for irregular singular connections (theorem 6.1) if we consider only degree. The proof follows Deligne's proof of the product formula in positive characteristic [12]. The missing ingredient is a Künneth formula for rapid decay cohomology.

Theorem 1.3. *Suppose that* U *and* V *are smooth complex analytic varieties. Let* $\boldsymbol{\mathcal{M}}$ *and* $\boldsymbol{\mathcal{N}}$ *be local systems on* U *and* V*, and suppose that* ϕ *(resp.* ψ*) is a regular function on* U *(resp.* V*). Then, there is a natural quasi-isomorphism*

$$R\Gamma_c(U \times V; \boldsymbol{\mathcal{M}} \boxtimes \boldsymbol{\mathcal{N}}, \phi + \psi) \cong R\Gamma_c(U; \boldsymbol{\mathcal{M}}, \phi) \boxtimes R\Gamma_c(V; \boldsymbol{\mathcal{M}}, \psi).$$

This theorem is a variation on the Thom-Sebastiani theorem, found in [22] Theorem 1.2.2. The version needed here is proved in section 2.7.

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2. Riemann Hilbert Correspondence

We begin with a review of the theory of algebraic $\mathcal{D}$-modules. The main references will be [1] and [15], but the material specific to $\mathcal{D}$-modules on curves is found in [19] and [20]. Throughout, k and M will be fields with fixed imbedding in $\mathbb{C}$; k will be field of definition for De Rham cohomology, and M will be the field of definition for moderate growth cohomology. Typically, k will be an algebraic extension of $\mathbb{Q}$, and $M = \mathbb{Q}(e^{2\pi\sqrt{-1}\alpha_1}, \ldots, e^{2\pi\sqrt{-1}\alpha_n})$ for $\alpha_i \in k$.

2.1. **Holonomic D-modules and Constructible Sheaves.** In this section, k will be an algebraically closed field with fixed imbedding $k \subset \mathbb{C}$. Let X/k be a smooth, quasi-projective algebraic variety, and suppose that $\mathscr{F}$ is a holonomic $\mathcal{D}_X$-module[5].

There is a dualizing sheaf $\mathcal{D}_X^\Omega$([1], Lecture 3, section 5), and we define

$$\mathbb{D}(\mathscr{F}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathscr{F}, \mathcal{D}_X^\Omega).$$

In general, $\mathbb{D}(\mathbb{D}(\mathscr{F})) \cong \mathscr{F}$ whenever $\mathscr{F}$ is coherent; however Roos's theorem (*ibid.*) implies that $\mathbb{D}(\mathscr{F})$ is concentrated in degree 0 if and only if $\mathscr{F}$ is holonomic. When $\mathscr{F}$ is an integrable connection, $\mathbb{D}(\mathscr{F}) \cong \mathscr{F}^\vee$, where $\mathscr{F}^\vee$ is the dual connection.

We will adopt the notation of [1] to describe inverse and direct images of $\mathcal{D}$-modules. Therefore, if $\phi : Y \to X$ is a morphism of varieties, we use $\phi^\cdot(\mathscr{F}) = \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathscr{F}$ to denote the standard pull-back for $\mathcal{D}$-modules. In the derived category,

$$\phi^!(\mathscr{F}) = L\phi^\cdot(\mathscr{F})[\dim(X) - \dim(Y)],$$
$$\phi^*(\mathscr{F}) = \mathbb{D}(\phi^!\mathbb{D}(\mathscr{F})),$$

and $\phi_!$ is the left (resp. ϕ_* is the right) adjoint of $\phi^!$ (resp. ϕ^*). When $X = \mathrm{Spec}(k)$, and $\mathscr{F}'$ is a holonomic $\mathcal{D}_Y$-module, we define

$$R\Gamma(Y; \mathscr{F}') = \phi_*\mathscr{F}' \quad \text{and}$$
$$R\Gamma_c(Y; \mathscr{F}') = \phi_!\mathscr{F}'.$$

When $\mathscr{F}$ is holonomic, there exists an open, dense subset $j : V \hookrightarrow X$ with the property that $j^*\mathscr{F}$ is $\mathcal{O}_V$-coherent ([2], lecture 2). Therefore, if $i : Y \hookrightarrow X$ is the complement of V, there is a distinguished triangle

$$i_*i^!\mathscr{F} \to \mathscr{F} \to j_*j^*\mathscr{F} \xrightarrow{[1]} \tag{2.1.1}$$

in the derived category of $\mathcal{D}_X$-modules with holonomic cohomology. In particular, $j^*\mathscr{F}$ is quasi-isomorphic to a vector bundle on V with algebraic connection. When X is a curve, $Y \cong \coprod \mathrm{Spec}(k)$ and $i^!\mathscr{F}$ is a direct sum of complexes of k-vector spaces.

The projection formula ([1], Lecture 3 section 12) for $\mathcal{D}$-modules states that

$$\phi_*(\mathscr{F}' \otimes^L_{\mathcal{O}_Y} \phi^!\mathscr{F}) \cong \phi_*(\mathscr{F}') \otimes^L_{\mathcal{O}_X} \mathscr{F}[\dim(Y) - \dim(X)].$$

Let X and Y be smooth varieties over k, and suppose that $\mathscr{F}$ is a $\mathcal{D}_Y$-module and $\mathscr{F}'$ is a $\mathcal{D}_X$-module. We denote the exterior tensor product of $\mathscr{F}$ and $\mathscr{F}'$ on $X \times Y$ by $\mathscr{F} \boxtimes \mathscr{F}'$ ([1], lecture 3.11). There is a Künneth formula for $\mathcal{D}$-modules:

[5] in the sense of [1], lecture 2 section 11. Unless specified, all $\mathcal{D}_X$-modules will be *left* $\mathcal{D}_X$-modules.

Proposition 2.1 (Künneth Formula). *There are natural isomorphisms*

$$R\Gamma(X \times Y; \mathscr{F}' \boxtimes \mathscr{F}) \cong R\Gamma(X; \mathscr{F}') \otimes_k R\Gamma(Y; \mathscr{F}). \tag{2.1.2}$$

and

$$R\Gamma_c(X \times Y; \mathscr{F}' \boxtimes \mathscr{F}) \cong R\Gamma_c(X; \mathscr{F}') \otimes_k R\Gamma(Y; \mathscr{F}'). \tag{2.1.3}$$

Proof. Equation (2.1.2) is proved by repeated application of the projection formula. The second equation follows from (2.1.2), and the observation that $\mathbb{D}(\mathscr{F}' \boxtimes \mathscr{F}) \cong \mathbb{D}(\mathscr{F}') \boxtimes \mathbb{D}(\mathscr{F})$. □

2.2. **Analytic and Formal $\mathcal{D}$-modules.** Let X^{an} be the complex analytic structure on $X(\mathbb{C})$. The pullback of $\mathscr{F}$ to X^{an} is an analytic $\mathcal{D}_{X^{an}}$-module $\mathscr{F}^{an}$. Notice that if $\mathscr{F}$ is an integrable connection, then $\mathscr{F}^{an}$ is an analytic vector bundle. Moreover, the horizontal sections define a local system $(\mathscr{F}^{an})^{\nabla}$ that generates $\mathscr{F}^{an}$ as a $\mathcal{O}_{X^{an}}$-module.

The De Rham functor allows us to compare holonomic $\mathcal{D}_X$-modules with perverse sheaves on X^{an}.

Definition 2.2 (De Rham Functor). *Let $\Omega^i_{X^{an}}$ be the sheaf of holomorphic forms on X^{an}, so the complex $\Omega^*_{X^{an}}$ with exterior differential d is the standard de Rham complex of X. Define*

$$\mathrm{DR}(\mathscr{F}) = \mathscr{F}^{an} \otimes^L_{\mathcal{O}_X} \Omega_{X^{an}}[\dim(X)].$$

Therefore, $\mathrm{DR}(\mathscr{F})$ *lies in the derived category of $\mathbb{C}_{X^{an}}$-modules.*

Recall (2.1.1). Since $j^*\mathscr{F}$ is isomorphic to a vector bundle with connection $\nabla_{j^*\mathscr{F}}$, $\mathrm{DR}(j^*\mathscr{F})$ is quasi-isomorphic to the perverse local system generated by horizontal sections.

Suppose that $\phi : Y^{an} \to X^{an}$ is a morphism of complex analytic manifolds. When $\boldsymbol{\mathcal{F}}$ is a complex of sheaves, we will use $\phi^{-1}\boldsymbol{\mathcal{F}}$ to denote the inverse image of $\boldsymbol{\mathcal{F}}$. Abusing notation, ϕ^* and ϕ_* will be used for the derived pull-back and push-forward in the analytic category, respectively, and $\phi^!$, $\phi_!$ will be used to denote pull-back and push-forward with compact support. We remark that the DR functor does not necessarily commute with push-forwards and pull-backs.

In addition to studying the analytic structure of $\mathcal{D}_X$-modules, we will also consider localization to a power series ring. Let C be a curve, and $x \in C$ a point. Let $\mathfrak{o} = \widehat{\mathcal{O}}_{C,x}$ be the ring of formal power series at $\mathcal{O}_X$, and let K be the field of laurent series at x. If $\mathscr{F}$ is a holonomic $\mathcal{D}_X$-module, then $\mathscr{F}_{\mathfrak{o}} = \mathscr{F} \hat{\otimes}_{\mathcal{O}_C} \mathfrak{o}$ and $\mathscr{F}_K = \mathscr{F} \hat{\otimes}_{\mathcal{O}_C} K$. $\mathscr{F}_K$ is a finite dimensional K-vector space. The following definition of regularity is equivalent to various other formulations that appear in the literature (Corollary 1.1.6, p. 47 [20]).

Definition 2.3. *Fix a parameter z, identifying $\mathfrak{o} \cong k[[z]]$ and $K \cong k((z))$. Then, we say that $\mathscr{F}$ (resp. $\mathscr{F}_o$, $\mathscr{F}_K$) is* regular singular *at x if there exists an $\mathfrak{o}$ submodule $\mathscr{F}' \subset \mathscr{F}_K$ with the property that*

$$z\frac{\partial}{\partial z}\mathscr{F}' \subset \mathscr{F}'.$$

If no such $\mathfrak{o}$-submodule exists, $\mathscr{F}$ is irregular singular*. We say that $\mathscr{F}$ is regular singular if regular at all singular points $x \in C$.*

Now suppose X/k is an n-dimensional smooth quasi-projective variety and $\mathscr{F}'$ is a $\mathcal{D}_X$-module $\mathscr{F}$ is regular singular if, for any inclusion $\iota : \mathrm{Spec}(K) \hookrightarrow V$, $\mathscr{V} \hat{\otimes}^{L}_{\mathcal{O}_X} K$ has regular singular cohomology.

For general $\mathscr{F}$, we can measure the defect from regularity at a singular point by the irregularity index $i_x(\mathscr{F})$. Like regularity, this property only depends on the formal completion of $\mathscr{F}$ at x. See [13], lemme 6.21 for a formal definition. For example, suppose that $\mathscr{F}$ is the trivial line bundle with connection $\nabla = d + \omega\wedge$. Then, $i_x(\mathscr{F}) = \max(0, -\mathrm{ord}_x(\omega) + 1)$. Note that there is another standard formulation of the irregularity index in the literature, described in terms of local Euler characteristics ([19], Chapter 4, theorem 4.1).

We define $\mathbf{D}^b_{hol}(\mathcal{D}_X)$ to be the bounded derived category of complexes of $\mathcal{D}_X$-modules with holonomic cohomology, and $\mathbf{D}^b_{RS}(\mathcal{D}_X)$ to be the full subcategory of complexes with regular singular cohomology. Furthermore, we define $\mathbf{D}^b_{con}(\mathbb{C}_{X^{an}})$ to be the bounded derived category of $\mathbb{C}_{X^{an}}$-modules that are constructible with respect to an algebraic stratification of X^{an}.

Theorem 2.4 (Riemann-Hilbert Correspondence ([1] Lecture 5, Main Theorem C))**.** *Let X/k and Y/k be a smooth algebraic varieties, and let $\phi : Y \to X$ be a morphism.*

1. $\mathrm{DR}(\mathbf{D}^b_{hol}(\mathcal{D}_X)) \subset \mathbf{D}^b_{con}(\mathbb{C}_{X^{an}})$.
2. *On the subcategories $\mathbf{D}^b_{RS}(\mathcal{D}_X)$ and $\mathbf{D}^b_{RS}(\mathcal{D}_Y)$,* DR *commutes with duality, ϕ_*, ϕ^*, $\phi_!$, and $\phi^!$.*
3. DR *defines an equivalence of categories between $\mathbf{D}^b_{RS}(\mathcal{D}_X)$ and $\mathbf{D}^b_{con}(\mathbb{C}_X)$. The restriction of* DR *to complexes of pure degree* 0 *defines an equivalence between holonomic $\mathcal{D}_X$-modules, and perverse sheaves on X^{an} (with respect to the middle perversity).*

2.3. **Non-characteristic Maps.** The Riemann-Hilbert correspondence fails for irregular singular $\mathcal{D}$-modules. Under certain conditions, the DR-functor is still compatible with pullbacks, the projection formula, and tensor products. However, the irregular singular case requires a careful analysis of the characteristic variety.

Let T^*X be the cotangent space to X, and $T^*_X X$ the zero section. Let $\phi : Y \to X$ be a morphism of smooth varieties. There are natural maps $T^*Y \xleftarrow{\rho_\phi} Y \times_X T^*X \xrightarrow{\varpi_\phi} T^*X$, and we define $T^*_Y X = \rho_\phi^{-1}(T^*_Y Y)$. Suppose that $\mathscr{F}$ is a $\mathcal{D}_X$-module, and $\boldsymbol{\mathcal{F}} \in \mathbf{D}^b_{con}(M_{X^{an}})$ for some field M. The characteristic variety (or singular support, [1] Lecture 2, section 8) of $\mathscr{F}$, written $\mathrm{Ch}(\mathscr{F})$, is a subvariety of T^*X that is invariant under homothety in the fiber above a point $x \in X$. Similarly, the micro-support ([16], proposition 5.1.1) of $\boldsymbol{\mathcal{F}}$, written $\mathrm{SS}(\boldsymbol{\mathcal{F}})$, is a subvariety of $T^*_{X^{an}}$.

We omit the precise definitions since they are standard but fairly technical; here it is sufficient to understand the smooth case. In particular, when $\mathscr{F}$ (resp. $\boldsymbol{\mathcal{F}}$) is an integrable connection, (resp. local system), then $\mathrm{Ch}(\mathscr{F}) = T^*_X X$ and $\mathrm{SS}(\boldsymbol{\mathcal{F}}) = T^*_{X^{an}} X^{an}$. We say that ϕ is non-characteristic with respect to $\mathscr{F}$, or $\boldsymbol{\mathcal{F}}$, if

$$\varpi_\phi^{-1}(\mathrm{Ch}(\mathscr{F})) \cap T^*_Y X \subset Y \times_X T^*_X X; \qquad \text{or,}$$
$$\varpi_\phi^{-1}(\mathrm{SS}(\boldsymbol{\mathcal{F}})) \cap T^*_{Y^{an}} X^{an} \subset Y^{an} \times_X T^*_{X^{an}} X^{an}.$$

Notice that whenever ϕ is a smooth map, or $\mathscr{F}$ is an integrable connection, ϕ is non-characteristic with respect to $\mathscr{F}$.

Theorem 2.5 ([15], theorems 2.4.6, 2.7.1; [16], proposition 5.4.13)**.** *Suppose that ϕ is non-characteristic with respect to $\mathscr{F}$ and $\boldsymbol{\mathcal{F}}$. Then, $L^i\phi^{\cdot}\mathscr{F}$ vanishes for $i \neq 0$, and $\mathbb{D}(L^i\phi^{\cdot}\mathscr{F}) \cong L^i\phi^{\cdot}(\mathbb{D}\mathscr{F})$. In particular,*

$$\phi^*(\mathscr{F}) \cong \phi^!(\mathscr{F})[2(\dim(Y) - \dim(X))] \quad \textit{and}$$
$$\phi^*(\boldsymbol{\mathcal{F}}) \cong \phi^!(\boldsymbol{\mathcal{F}})[2(\dim(Y) - \dim(X))].$$

When ϕ is non-characteristic with respect to $\mathscr{F}$, we write $\phi^\Delta = \phi^{\cdot}$. When $\boldsymbol{\mathcal{F}}$ is a complex of constructible sheaves, we write $\phi^\Delta = \phi^*[\dim(X) - \dim(Y)]$.

Let $\Delta_X : X \to X \times X$ be the diagonal map. $T^*_X(X \times X)$ is the conormal bundle to the diagonal imbedding of X, so the fiber above (x, x) is the set $\{(\xi, \zeta) \in (T^*_x X)^2 : \xi + \zeta = 0\}$. There is an isomorphism $\alpha : T^*_X(X \times X) \to T^*_X(X \times X)$ given locally by $\alpha(x, \xi) = ((x, x), (\xi, -\xi))$.

If $\mathscr{F}$ and $\mathscr{G}$ are $\mathcal{D}_X$-modules, $\mathrm{Ch}(\mathscr{F} \boxtimes \mathscr{G}) = \mathrm{Ch}(\mathscr{F}) \times \mathrm{Ch}(\mathscr{G})$. Since characteristic varieties are invariant under homothety (in this case, multiplication by -1 in the fibers),

$$\varpi^{-1}_{\Delta_X}(\mathrm{Ch}(\mathscr{F} \boxtimes \mathscr{G})) \cap T^*_X(X \times X) = \alpha(\mathrm{Ch}(\mathscr{F}) \cap \mathrm{Ch}(\mathscr{G})).$$

Therefore, Δ_X is non-characteristic for $\mathscr{F}$ and $\mathscr{G}$ if and only if $\mathrm{Ch}(\mathscr{F}) \cap \mathrm{Ch}(\mathscr{G}) \subset T^*_X X$. In this case,

$$\Delta^*_X(\mathscr{F} \boxtimes \mathscr{G}) \cong \mathscr{F} \otimes^L_{\mathcal{O}_X} \mathscr{G}[\dim(X)].$$

Proposition 2.6 (Non-characteristic projection formula)**.** *Suppose that $\phi : Y \to X$, $\mathscr{F}$ is a holonomic $\mathcal{D}_X$-module, and $\mathscr{F}'$ is a holonomic $\mathcal{D}_Y$-modules. If $\mathrm{Ch}(\phi_!\mathscr{F}') \cap \mathrm{Ch}(\mathscr{F}) \subset T^*_X X$ and $\mathrm{Ch}(\mathscr{F}') \cap \mathrm{Ch}(\pi^\Delta\mathscr{F}) \subset T^*_Y Y$, then*

$$\phi_!(\phi^\Delta(\mathscr{F}) \otimes^L_{\mathcal{O}_Y} \mathscr{F}') \cong \mathscr{F} \otimes^L_{\mathcal{O}_X} \phi_!\mathscr{F}'.$$

Proof. Let $\Gamma_\phi : Y \to Y \times X$ be the graph of ϕ. Then, by above, $\Gamma^*_\phi(\mathscr{F}' \boxtimes \mathscr{F}) \cong \mathscr{F}' \otimes^L_{\mathcal{O}_X} \phi^*\mathscr{F}$. The result follows by applying base change to the diagram

$$\begin{tikzcd} Y \arrow[r, "\Gamma_\phi"] \arrow[d, "\phi"] & Y \times X \arrow[d, "\mathrm{id}\times\phi"] \\ X \arrow[r, "\Delta_X"] & X \times X. \end{tikzcd}$$

$\square$

Finally, non-characteristic maps behave well with respect to the DR functor.

Proposition 2.7 ([15], proposition 4.7.6)**.** *If ϕ is non-characteristic for $\mathscr{F}$, then there are natural isomorphisms*

$$\mathrm{DR}(f^*\mathscr{F}) \cong f^*\mathrm{DR}(\mathscr{F}) \quad \textit{and}$$
$$\mathrm{DR}(f^!\mathscr{F}) \cong f^!\mathrm{DR}(\mathscr{F}).$$

2.4. **Stokes Filtrations.** When X is a curve, the Riemann-Hilbert correspondence generalizes to a correspondence between holonomic $\mathcal{D}_X$-modules and perverse sheaves with a stokes filtration. Let $i : Y \hookrightarrow X$ be a reduced divisor, and $j : V \hookrightarrow X$ be the complement. We define $\pi : \widetilde{X}^{an} \to X^{an}$ to be the real oriented

blow-up of X^{an} along Y; the diagram is

$$\begin{tikzcd} V^{an} \arrow[r, "\tilde{j}"] \arrow[d, equal] & \widetilde{X}^{an} \arrow[d, "\pi"] & \widetilde{Y} \arrow[l, "\tilde{i}"'] \arrow[d, "\pi|_Y"] \\ V^{an} \arrow[r, "j"] & X^{an} & Y. \arrow[l, "i"'] \end{tikzcd}$$

Above, $\widetilde{Y} \cong \coprod_{y\in Y} S^1$.

Fix a parameter z vanishing at some $y \in Y$, and let $S^1_y = \pi^{-1}(y)$. We define a rank $n-p$ local system $\Omega_{p,n}(y)$ on S^1_y by

$$\Omega_{p,n}(y) = \{\sum_{i=-n}^{\infty} a_i z^{i/p} dz; a_i \in \mathbb{C}\}/\{\sum_{j/p\geq -1}^{\infty} b_j z^{j/p} dz; b_j \in \mathbb{C}\}.$$

At each $\theta \in S^1_y$, there is a partial ordering $\leq_\theta$ on the stalk $\Omega_{p,n}(y)_\theta$ determined by $\omega \leq_\theta \eta$ whenever $|e^{\int \omega - \eta}|$ has moderate growth near θ.

Let $\mathscr{E}$ be a vector bundle on V with a $\mathcal{D}_V$-module structure and $\boldsymbol{\mathcal{E}} = \mathrm{DR}(\mathscr{E})$. By [19], theorem 2.2, there is a canonical Stokes filtration $(\boldsymbol{\mathcal{E}}^\nu_y)_{\nu\in\Omega_{p,n}(y)}$ on $\tilde{j}_*\boldsymbol{\mathcal{E}}|_{S^1_y}$. Furthermore, there are subsheaves $\boldsymbol{\mathcal{E}}^0 \subset \tilde{j}_*\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{E}}^{<0} \subset \boldsymbol{\mathcal{E}}$ that satisfy $\boldsymbol{\mathcal{E}}^0|_{S^1_y} = \boldsymbol{\mathcal{E}}^0_y$, $\boldsymbol{\mathcal{E}}^{<0}|_{S^1_y} = \sum_{\substack{\eta\leq 0\\ \eta\neq 0}} \boldsymbol{\mathcal{E}}^\eta_\theta$, and $\tilde{j}^*\boldsymbol{\mathcal{E}}^0 \cong \tilde{j}^*\boldsymbol{\mathcal{E}}^{<0} \cong \boldsymbol{\mathcal{E}}$.

Theorem 2.8. *[[19], 4.3, Theorems 3.1, 3.2] There are canonical isomorphisms*

$$\mathrm{DR}(j_*\mathscr{E}) \cong R\pi_*(\boldsymbol{\mathcal{E}}^0)$$

and

$$\mathrm{DR}(j_!\mathscr{E}) \cong R\pi_*(\boldsymbol{\mathcal{E}}^{<0}). \tag{2.4.1}$$

Corollary 2.8.1. *Let $H^*_{\mathrm{DR}}(V;\mathscr{E})$ be the algebraic de Rham cohomology of $\mathscr{E}$. Then, there is a canonical isomorphism*

$$H^*_{\mathrm{DR}}(V;\mathscr{E}) \cong H^*(\widetilde{X};\boldsymbol{\mathcal{E}}^0).$$

We call $H^*(\widetilde{X};\boldsymbol{\mathcal{E}}^0)$ the 'moderate growth cohomology' of $\boldsymbol{\mathcal{E}}$ with stokes filtration $\{\boldsymbol{\mathcal{E}}^\omega\}$.

2.5. **Elementary $\mathcal{D}_X$-modules.** We will consider a class of holonomic $\mathcal{D}$-modules for which theorem 2.8 admits a simple description. Let $\mathbb{A}^1 = \mathrm{Spec}(k[t])$, and let $\mathcal{O}_{dt}$ be the $\mathcal{D}_{\mathbb{A}^1}$-module defined by $\mathcal{D}_{\mathbb{A}^1}/\mathcal{D}_{\mathbb{A}^1}(\frac{\partial}{\partial t} - 1)$. The following definition comes from [21]:

Definition 2.9 (Elementary $\mathcal{D}$-modules)**.** *Let V/k be a smooth, quasi-projective variety (for now, we put no conditions on $\dim_k(V)$), and let $\mathscr{M}$ be a regular holonomic $\mathcal{D}_V$-module. Given a regular function $\phi : V \to \mathbb{A}^1$, we define the elementary $\mathcal{D}$-module $\mathscr{M}_{d\phi}$ by*

$$\mathscr{M}_{d\phi} = \mathscr{M} \otimes_{\mathcal{O}_V} \phi^\Delta \mathcal{O}_{dt}.$$

We call ϕ the 'Morse' function for $\mathscr{E}$.

Proposition 2.10. *If $\mathscr{M}_{d\phi}$ is an elementary $\mathcal{D}_V$-module, then*

$$\mathbb{D}(\mathscr{M}_{d\phi}) \cong (\mathbb{D}(\mathscr{M}))_{-d\phi}.$$

Proof. Let $\mathcal{D}_V^\omega$ be the dualizing sheaf for $\mathcal{D}_V$-modules ([1], 3.5). Taking a resolution of $\mathscr{M}$ by projective $\mathcal{D}_V$-modules, it suffices to show that $\mathcal{H}om_{\mathcal{D}_V}(\mathcal{D}_V \otimes_{\mathcal{O}_V} \mathcal{O}_{d\phi}, \mathcal{D}_V^\omega) \cong \mathcal{D}_V^\omega \otimes_{\mathcal{O}_V} \mathcal{O}_{-d\phi}$. Since $\mathcal{O}_{d\phi}$ is $\mathcal{O}_V$- coherent, $\mathbb{D}(\mathcal{O}_{d\phi}) \cong \mathcal{O}_{d\phi}^\vee \cong \mathcal{O}_{-d\phi}$. Furthermore, $\mathbb{D}(\mathcal{O}_{d\phi}) \otimes_{\mathcal{O}_V} \mathcal{O}_{d\phi} \cong \mathcal{O}_V$.

Therefore,

$$\mathcal{H}om_{\mathcal{D}_V}(\mathcal{D}_V \otimes_{\mathcal{O}_V} \mathcal{O}_{d\phi}, \mathcal{D}_V^\omega) \cong \mathcal{H}om_{\mathcal{D}_V}(\mathcal{D}_V, \mathcal{D}_V^\omega \otimes_{\mathcal{O}_V} \mathcal{O}_{-d\phi}) \cong \mathcal{D}_V^\omega \otimes_{\mathcal{O}_V} \mathcal{O}_{-d\phi}. \tag{2.5.1}$$

□

Let $\widetilde{\mathbb{P}}^1 \xrightarrow{\pi} \mathbb{P}^1$ be the real oriented blow-up of $\mathbb{P}^1$ at ∞, and $j : \mathbb{A}^1 \to \widetilde{\mathbb{P}}^1$. Define $I \subset \pi^{-1}(\infty)$ to be the interval on which $dt \leq_\theta 0$,[6] and $\widetilde{\mathbb{P}}^1_I = j(\mathbb{A}^1) \cup I$. Let $\alpha : \mathbb{A}^1 \hookrightarrow \widetilde{\mathbb{P}}^1_I$ and $\beta : \widetilde{\mathbb{P}}^1_I \hookrightarrow \widetilde{\mathbb{P}}^1$. If $\boldsymbol{\mathcal{O}}_{dt} = \mathrm{DR}(\mathcal{O}_{dt})$, then the 0-filtered component of the corresponding Stokes filtration is given by

$$\boldsymbol{\mathcal{O}}_{dt}^0 = \beta_! \alpha_* \boldsymbol{\mathcal{O}}_{dt}.$$

In this case, $\boldsymbol{\mathcal{O}}_{dt}^{<0} = \beta_! \alpha_* \boldsymbol{\mathcal{O}}_{dt}$ as well.

Now, let $\mathscr{M}_{d\phi}$ be an elementary $\mathcal{D}_V$-module. Suppose that X is a projective closure of V with a morphism $\Phi : X \to \mathbb{P}^1$, with the property that $\Phi|_V = \phi$. We define $\widetilde{X} = X^{an} \times_\Phi \widetilde{\mathbb{P}}^1$ and $\widetilde{X}_I = X^{an} \times_\Phi \widetilde{\mathbb{P}}^1_I$. There are inclusions

$$V^{an} \xrightarrow{\alpha_X} \widetilde{X}_I \xrightarrow{\beta_X} \widetilde{X}.$$

Define $\tilde{j}_X = \beta_X \circ \alpha_X$ and $\boldsymbol{\mathcal{M}}_{d\phi} = \mathrm{DR}(\mathscr{M}_{d\phi})$, and $j_X : V \to X$. When X is a curve, $\widetilde{X}$ is homeomorphic to the real analytic blow-up of X^{an} along the divisor $\phi^{-1}(\infty)$.

Observe that α_X may be factored as $\alpha_2 \circ \alpha_1$, where $\alpha_1 : V^{an} \to X \times_\Phi \mathbb{A}^1$ and $\alpha_2 : X \times_\Phi \mathbb{A}^1 \to \widetilde{X}_I$. In particular, $X \times_\Phi \mathbb{A}^1 \to \mathbb{A}^1$ is a proper map.

Definition 2.11. *Let $\phi : V \to \mathbb{A}^1$ be a regular function, and let $\boldsymbol{\mathcal{M}}$ be a complex of sheaves on V^{an}. Define*

$$R\Gamma(V; \boldsymbol{\mathcal{M}}, \phi) = R\Gamma(\widetilde{X}^{an}; (\beta_X)_! (\alpha_X)_* \boldsymbol{\mathcal{M}}),$$

with X, α_X and β_X as defined above. Moreover, define

$$R\Gamma_c(V; \boldsymbol{\mathcal{M}}, \phi) = R\Gamma(\widetilde{X}^{an}; (\beta_X)_! (\alpha_2)_* (\alpha_1)_! \boldsymbol{\mathcal{M}}).$$

Theorem 2.12. *Suppose that $\mathscr{M}_{d\phi}$ is an elementary $\mathcal{D}_V$-module, and $\boldsymbol{\mathcal{M}}_{d\phi} = \mathrm{DR}(\mathscr{M}_{d\phi})$. There are canonical isomorphisms*

$$R\Gamma(V; \mathscr{M}_{d\phi}) \cong R\Gamma(\widetilde{X}; (\beta_X)_! (\alpha_X)_* \boldsymbol{\mathcal{M}}_{d\phi}) \tag{2.5.2}$$

and

$$R\Gamma_c(V; \mathscr{M}_{d\phi}) \cong R\Gamma(\widetilde{X}; (\beta_X)_! (\alpha_1)_* (\alpha_2)_! \boldsymbol{\mathcal{M}}_{d\phi}). \tag{2.5.3}$$

The first part is theorem 1.1 in [20]. We include a key step that allows us to reduce to the setting of theorem 2.8.

Lemma 2.13. *There are natural isomorphisms*

$$R\Gamma(V; \mathscr{M}_{d\phi}) \cong R\Gamma(\mathbb{A}^1; (\phi_* \mathscr{M})_{dt}) \quad \textit{and}$$
$$R\Gamma(V^{an}; \boldsymbol{\mathcal{M}}, \phi) \cong R\Gamma(\mathbb{A}^1; (\phi_* \boldsymbol{\mathcal{M}}), t).$$

[6] If $z = 1/t$ is the parameter at ∞, this is equivalent to $-\frac{dz}{z^2} \leq_\theta 0$

Proof. By the projection formula for $\mathcal{D}$-modules,

$$\phi_*\mathscr{M}_{d\phi} \cong (\phi_*\mathscr{M}) \otimes_{\mathcal{O}_{\mathbb{A}^1}} \mathcal{O}_{dt}.$$

On the other hand, since Φ is a proper map,

$$\Phi_*(\beta_X)_!(\alpha_X)_*\boldsymbol{\mathcal{M}}_{d\phi} \cong \beta_!\alpha_*\phi_*\boldsymbol{\mathcal{M}}_{d\phi}.$$

Since $\mathscr{M}$ is regular singular, the sections of $H^i\mathrm{DR}(\phi_*\mathscr{M})$ have moderate growth at infinity, and the sections of $H^i\mathrm{DR}((\phi_*\mathscr{M}) \otimes_{\mathcal{O}_{\mathbb{A}^1}} \mathcal{O}_{dt})$ have moderate growth on the sector I. Theorem 2.4 implies that $\mathrm{DR}(\phi_*\mathscr{M}) \cong \phi_*\mathrm{DR}(\mathscr{M})$. It follows that $\phi_*\boldsymbol{\mathcal{M}}_{d\phi} \cong \mathrm{DR}((\phi_*\mathscr{M})_{dt})$. □

Here, we show that the first statement of theorem 2.12, (2.5.2), implies the second, (2.5.3).

Proof. In the second case,

$$\begin{aligned}
\phi_!\mathscr{M}_{d\phi} &\cong \mathbb{D}\phi_*\mathbb{D}(\mathscr{M}_{d\phi}) \\
&\cong \mathbb{D}\phi_*\left((\mathbb{D}\mathscr{M})_{-d\phi}\right) \quad \text{by proposition 2.10} \\
&\cong \mathbb{D}\left((\phi_*\mathbb{D}\mathscr{M})_{-dt}\right) \\
&\cong \left(\phi_!\mathscr{M}\right)_{dt}.
\end{aligned}$$

Similarly, if $\Phi' : X \times_\Phi \mathbb{A}^1 \to \mathbb{A}^1$,

$$\begin{aligned}
\Phi_!(\beta_X)_!(\alpha_1)_*(\alpha_2)_!\boldsymbol{\mathcal{M}}_{d\phi} &\cong (\beta)_!(\alpha)_*\Phi'_!(\alpha_2)_!\boldsymbol{\mathcal{M}}_{d\phi} \\
&\cong (\beta)_!(\alpha)_*\phi_!\boldsymbol{\mathcal{M}}_{d\phi}.
\end{aligned}$$

As before, $\mathrm{DR}(\phi_!\mathscr{M}) \cong \phi_!\mathrm{DR}(\mathscr{M})$.

□

2.6. **Betti structures.**

Definition 2.14 (Betti Structure)**.** *Let $\mathscr{M}$ be a holonomic $\mathcal{D}_X$-module, and let $M \subset \mathbb{C}$ be a field with fixed complex imbedding. A Betti structure for $\mathscr{M}$ is a perverse sheaf of M_X-modules $\boldsymbol{\mathcal{M}}$ with the property that $\boldsymbol{\mathcal{M}} \otimes_M \mathbb{C}_X \cong \mathrm{DR}(\mathscr{M})$.*

We define $\mathrm{MB}(\mathcal{D}_X, M)$ to be the category of pairs $(\mathscr{M}, \boldsymbol{\mathcal{M}})$ consisting of: a holonomic $\mathcal{D}_X$-modules $\mathscr{M}$, Betti structure $\boldsymbol{\mathcal{M}}$ that has coefficients in M, and a fixed compatibility isomorphism $\alpha : \boldsymbol{\mathcal{M}} \otimes_M \mathbb{C} \to \mathrm{DR}(\mathscr{M})$. A morphism in $\mathrm{MB}(\mathcal{D}_X, M)$ is given by a pair of maps $\psi_k : \mathscr{M} \to \mathscr{M}'$ and $\psi_M : \boldsymbol{\mathcal{M}} \to \boldsymbol{\mathcal{M}}'$ with the property that the following diagram commutes:

$$\begin{array}{ccc}
\boldsymbol{\mathcal{M}} & \xrightarrow{\alpha} & \mathrm{DR}(\mathscr{M}) \\
\downarrow{\scriptstyle \psi_M} & & \downarrow{\scriptstyle \mathrm{DR}(\psi_k)} \\
\boldsymbol{\mathcal{M}}' & \xrightarrow{\alpha'} & \mathrm{DR}(\mathscr{M}').
\end{array}$$

Therefore, two Betti structures are isomorphic if (ψ_k, ψ_M) are isomorphisms. Notice that Betti structures are well defined in the derived category: if $\mathscr{M}^*$ is a complex of $\mathcal{D}_X$-modules with holonomic cohomology, then $\boldsymbol{\mathcal{M}}^*$ is a complex of sheaves with a natural quasi-isomorphism $\boldsymbol{\mathcal{M}}^* \otimes_M \mathbb{C} \cong \mathrm{DR}(\mathscr{M})$. Thus, we define $\mathbf{D}^{\mathrm{b}}(\mathrm{MB})(\mathcal{D}_X, M)$ to be the category of pairs of complexes $(\mathscr{M}^*, \boldsymbol{\mathcal{M}}*)$, with $\mathscr{M} \in \mathbf{D}^b(\mathcal{D}_X)$ and $\boldsymbol{\mathcal{M}} \in \mathbf{D}^b(M_X)$, and a quasi-isomorphism $\alpha : \boldsymbol{\mathcal{M}} \otimes_M \mathbb{C} \cong \mathrm{DR}(\mathscr{M})$. For instance, we may define $(\mathscr{M}, \boldsymbol{\mathcal{M}})[n] = (\mathscr{M}[n], \boldsymbol{\mathcal{M}}[n])$.

Suppose $\mathscr{M}$ is non-singular on $V \subset X$. Fix a basepoint $x \in V(k)$ corresponding to a maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_V$. Let $k(x) = \mathcal{O}_V/\mathfrak{m}_x$ and $\mathbb{C}(x) = k(x) \otimes_k \mathbb{C}$. The fundamental group of V^{an} acts on $\mathrm{DR}(\mathscr{M})_x$, so it is a necessary condition that M contain the matrix coefficients of this representation. Furthermore, if there is a stokes filtration on $\tilde{j}_*\mathrm{DR}(\mathscr{M})$, M must be large enough so that there is a corresponding filtration on $\tilde{j}_*\boldsymbol{\mathcal{M}}$. By theorem 2.12, the second condition is always satisfied when $\mathscr{M}$ is an elementary $\mathcal{D}_V$-module.

Let i_x denote the inclusion of x. By theorem 2.4, $\mathrm{DR}(\mathscr{M})_x \cong \mathrm{DR}(i_x^*\mathscr{M}) \cong i_x^*\mathscr{M} \otimes_k \mathbb{C}$. If $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{M}}'$ are isomorphic Betti structures for $\mathscr{M}$, then there is a commutative diagram

$$\begin{tikzcd} \boldsymbol{\mathcal{M}}_x \arrow[r, "\iota"] \arrow[d, "(\psi_M)_x"] & i_x^*\mathscr{M} \otimes_k \mathbb{C} \arrow[d, "i_x^*\psi_k"] \\ \boldsymbol{\mathcal{M}}'_x \arrow[r, "\iota'"] & i_x^*\mathscr{M} \otimes_k \mathbb{C} \end{tikzcd}$$

Above, ι (resp ι') is defined by the composition

$$\boldsymbol{\mathcal{M}}_x \to \mathrm{DR}(\mathscr{M})_x \cong i_x^*\mathscr{M} \otimes_k \mathbb{C}. \tag{2.6.1}$$

Proposition 2.15. *Suppose $\mathscr{L}$ is a line bundle with integrable connection. Let $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\mathcal{L}}'$ be Betti structures for $\mathrm{DR}(\mathscr{L})$ with coefficients in M. Then, $(\mathscr{L}, \boldsymbol{\mathcal{L}}) \cong (\mathscr{L}, \boldsymbol{\mathcal{L}}')$ if and only if $\iota(\boldsymbol{\mathcal{L}}) = \alpha\iota'(\boldsymbol{\mathcal{L}}')$ with $\alpha \in k^\times$.*

Proof. Recall that $R\mathrm{Hom}_{\mathcal{D}_V}(\mathscr{M}, \mathscr{N}) \cong H^*_{\mathrm{DR}}(V; \mathbb{D}(\mathscr{M}) \otimes^L_{\mathcal{O}_X} \mathscr{N}[-\dim V])$ ([1], lecture 3, section 11). Since $\mathbb{D}(\mathscr{L}) \cong \mathscr{L}^\vee$, the dual line bundle with dual connection, it follows that

$$\mathrm{Hom}_{\mathcal{D}_V}(\mathscr{L}, \mathscr{L}) \cong H^0_{\mathrm{DR}}(\mathcal{O}_V[-\dim(V)]) \cong k.$$

Therefore, multiplication by α lifts to a global automorphism $\psi_\alpha \in \mathrm{Aut}_{\mathcal{D}_V}(\mathscr{L})$. $\square$

Suppose that $(\mathscr{M}, \boldsymbol{\mathcal{M}})$ and $(\mathscr{N}, \boldsymbol{\mathcal{N}})$ are in $\mathrm{MB}(\mathcal{D}_X, M)$. If $\mathrm{Ch}(\mathscr{M}) \cap \mathrm{Ch}(\mathscr{N}) \subset T^*_X X$, then the tensor product

$$(\mathscr{M}, \boldsymbol{\mathcal{M}}) \otimes (\mathscr{N}, \boldsymbol{\mathcal{N}}) = (\mathscr{M} \otimes \mathscr{N}, \boldsymbol{\mathcal{M}} \otimes \boldsymbol{\mathcal{N}})$$

is well defined by proposition 2.7, since the diagonal map Δ_X is non-characteristic with respect to $(\mathscr{M} \boxtimes \mathscr{N})$.

Definition 2.16 (Tate Twist)**.** *Let $\mathscr{M}$ be a $\mathcal{D}_V$-module, and $\boldsymbol{\mathcal{M}}$ a Betti structure for $\mathscr{M}$. For any integer n, define $\boldsymbol{\mathcal{M}}(n)$ to be the sheaf $[\boldsymbol{\mathcal{M}}(n)](V) = \left[(2\pi\sqrt{-1})^{-n}\boldsymbol{\mathcal{M}}(V)\right]$, for any open $V \subset V$. Furthermore, define $(\mathscr{M}, \boldsymbol{\mathcal{M}})(n) = (\mathscr{M}, \boldsymbol{\mathcal{M}}(n))$.*

For example, if $(2\pi\sqrt{-1})$ is transcendental over k and M, $(\mathscr{M}, \boldsymbol{\mathcal{M}})$ is not isomorphic to $(\mathscr{M}, \boldsymbol{\mathcal{M}})(1)$ even though $\boldsymbol{\mathcal{M}} \cong \boldsymbol{\mathcal{M}}(1)$ as sheaves.

By proposition 2.7, if $\phi : Y \to X$ is a map of smooth varieties that is non-characteristic for $\mathscr{M}$, then we may use ϕ^*, $\phi^!$, and ϕ^Δ to pull $(\mathscr{M}, \boldsymbol{\mathcal{M}})$ back to an element of $\mathrm{MB}(\mathcal{D}_Y, M)$.

Proposition 2.17. *Let $(\mathscr{M}, \boldsymbol{\mathcal{M}}) \in \mathrm{MB}(\mathcal{D}_V, M)$.*

(1) *Let $i : Y \to V$ be a closed imbedding, and $d_i = \dim(Y) - \dim(V)$. Furthermore, suppose that i is non-characteristic with respect to $\mathscr{M}$. Then, there is a commutative diagram*

$$i^*(\mathscr{M}, \boldsymbol{\mathcal{M}})[-d_i](-d_i) \cong i^!(\mathscr{M}, \boldsymbol{\mathcal{M}})[d_i]$$

(2) *If* $p : W \to V$ *is a smooth map, and* $d_p = \dim(W) - \dim(V)$,

$$p^!(\mathscr{M}, \boldsymbol{\mathcal{M}}) \cong p^*(\mathscr{M}, \boldsymbol{\mathcal{M}})[2d_p](d_p).$$

Proof. Observe that $i^!\mathscr{M} \cong \mathscr{M} \otimes_{\mathcal{O}_V} \mathcal{O}_Y[d_i]$. Since $i^*\mathscr{M} \cong \mathscr{M} \otimes_{\mathcal{O}_V} \mathcal{O}_Y[-d_i]$, we take ψ_k to be the identity map on $\mathscr{M} \otimes_{\mathcal{O}_V} \mathcal{O}_Y[d_i]$ and $\mathrm{DR}(\psi_k) = \psi_k \otimes_k \mathbb{C}$.

We first consider the case where Y has codimension 1. Let $j : V' \to V$ be the open complement of Y in V. Suppose that $\mathscr{M}$ is the trivial connection $\mathcal{O}_V$, $M = \mathbb{Q}$, and $\boldsymbol{\mathcal{M}}$ is the constant sheaf $\mathbb{Q}_V[\dim(V)]$. Let $\Omega^*_V(Y)$ be the complex of differential forms with log poles at Y. Then, there is a quasi-isomorphism of triangles

$$\begin{tikzcd}
\Omega^*_V[\dim(V)] \arrow[r] \arrow[d, "\sim"] & \Omega^*_V(Y)[\dim(V)] \arrow[r, "\frac{1}{2\pi\sqrt{-1}}\mathrm{Res}"] \arrow[d, "\sim"] & i_*\Omega^*_Y[\dim(Y)] \arrow[d, "\sim"] \\
\mathrm{DR}(\mathcal{O}_V) \arrow[r] & \mathrm{DR}(j_*j^*\mathcal{O}_V) \arrow[r] & \mathrm{DR}(i_*i^!\mathcal{O}_V)[-1]
\end{tikzcd}$$

where Res is the residue map. If f is a local defining function for Y, ψ_k has the following local description:

$$\psi_k : i^*\Omega^*_V[\dim(V)] \to \Omega^*_Y[\dim(Y)]$$
$$\omega \mapsto \frac{1}{2\pi\sqrt{-1}}\mathrm{Res}(\omega \wedge \frac{df}{f}).$$

Now, let T_ϵ be a tubular neighborhood of Y in V^{an}, let S_ϵ be the boundary, and let $T^\times_\epsilon = T_\epsilon \backslash Y$. There is a natural projection $\pi_\epsilon : T_\epsilon \to Y$, and inclusion $j_\epsilon : T^\times_\epsilon \to T_\epsilon$. It follows that there is an exact triangle

$$R\pi_{\epsilon *}(\mathbb{Q}_{T_\epsilon}) \to R\pi_{\epsilon *}(Rj_{\epsilon *}\mathbb{Q}_{T^\times_\epsilon}) \to i^!\mathbb{Q}_V;$$

in particular, $H^1R\pi_{\epsilon *}(Rj_{\epsilon *}\mathbb{Q}_{T^\times_\epsilon})[-1] \cong i^!\mathbb{Q}(V)$. Furthermore, since S_ϵ is a homotopy retract of $T^\times_\epsilon$, there is a quasi-isomorphism

$$\mathrm{Res} : H^1R\pi_{\epsilon *}(\mathbb{Q}_{S_\epsilon})[-1] \to i^!\mathbb{Q}(V).$$

Finally, the cup product with the Thom class maps $i^*\mathbb{Q}_V$ to $H^1R\pi_{\epsilon *}(\mathbb{Q}_{S_\epsilon})$. We define $\psi_M : i^*\mathbb{Q}_V[\dim(V)] \to i^!\mathbb{Q}_V(-1)[\dim(V)]$ to be the composition of the Thom isomorphism with $\frac{1}{2\pi\sqrt{-1}}\mathrm{Res}$.

By integration along the fiber of $S_\epsilon \to Y$, the following diagram commutes:

$$\begin{tikzcd}
i^*\mathbb{Q}_V[\dim(V)] \arrow[r] \arrow[d, "\psi_M"] & \mathrm{DR}(i^*\mathcal{O}_V) \arrow[d, "\mathrm{DR}(\psi_k)"] \\
i^!\mathbb{Q}_V[\dim(V)](-1) \arrow[r] & \mathrm{DR}(i^!\mathcal{O}_V).
\end{tikzcd} \tag{2.6.2}$$

Now, suppose that $\mathscr{M}$ is a vector bundle with connection and $\boldsymbol{\mathcal{M}}$ is a DR-sheaf for $\mathscr{M}$ with coefficients in M. Let F^* be a c-soft resolution of $\mathbb{Q}_V$. Then, $F^* \otimes_\mathbb{Q} \boldsymbol{\mathcal{M}}$ is a c-soft resolution of $\boldsymbol{\mathcal{M}}$ ([16] Lemma 2.5.12) It follows that there is a natural isomorphism $i^!\boldsymbol{\mathcal{M}} \cong (i^!\mathbb{Q}_V) \otimes_\mathbb{Q} \boldsymbol{\mathcal{M}}$. Similarly, $\mathscr{M}^{an} \cong \mathcal{O}^{an}_V \otimes_M \boldsymbol{\mathcal{M}}$. We define ψ_M and $\mathrm{DR}(\psi_k)$ by tensoring the maps in (2.6.2) with $\boldsymbol{\mathcal{M}}$.

When Y has codimension greater than one, Y is locally a complete intersection. Therefore, shrinking V if necessary, it is possible to construct a stratification $W = Y^0 \supset Y^1 \supset Y^2 \supset \ldots \supset Y^r = Y$, with inclusions $i_k : Y^k \to Y^{k-1}$. Part 1 of the proposition follows by induction on the codimension of Y.

We consider the case of a smooth map p. In this case, $p^!\boldsymbol{\mathcal{M}} \cong p^*\boldsymbol{\mathcal{M}} \otimes_{\mathbb{Q}} p^!\mathbb{Q}_W$ ([16], proposition 3.3.2), and $p^!\mathscr{M} \cong p^*\mathscr{M}[2d_p]$ ([1], Lecture 3.13). Therefore, it suffices to show that $p^!\mathbb{Q}_V \cong p^*\mathbb{Q}_V(d_p)[2d_p]$ as DR sheaves for $p^!\mathcal{O}_V$. We reduce to the case where p is a projection by considering the graph morphism

$$\begin{tikzcd} W \arrow[r, "\Gamma_p"] \arrow[dr, "p"'] & W \times V \arrow[d, "\mathrm{pr}_2"] \\ & V. \end{tikzcd}$$

If $\mathrm{pr}_2^!\mathbb{Q}_V \cong \mathrm{pr}_2{}^*\mathbb{Q}_V(\dim(W))[2\dim(V)]$, then the result follows by applying part 1 to Γ_p.

Assume $W \cong F \times V$, and p is the second projection. Fix a point $w \in F$, and let $i_v : V \to W$ be the map $i_w(v) = (w, v)$. Then, $i_w^! p^!\mathbb{Q}_U \cong i_w^* p^!\mathbb{Q}_U(-d_p)[-2d_p]$ by part 1. However, by composition of functors, $i_w^! p^!$ is naturally isomorphic to the identity. Therefore, $(p^!\mathbb{Q}_V)_{(w,v)} \cong (p^*\mathbb{Q}_V(d_p)[2d_p])_{(w,v)}$. By proposition 2.15, $p^!\mathbb{Q}_V$ is isomorphic to the Betti structure $p^*\mathbb{Q}_V(d_p)[2d_p]$. This proves part 2. $\square$

Define $\mathrm{MB}_{\mathrm{el}}(\mathcal{D}_X, M)$ to be the category of elementary holonomic $\mathcal{D}$-modules with Betti structure. Therefore, an element of $\mathrm{MB}_{\mathrm{el}}(\mathcal{D}_X, M)$ is given by the triple $(\mathscr{M}_{d\phi}, \boldsymbol{\mathcal{M}}_{d\phi}, \phi)$. Define

$$R\Gamma(X; (\mathscr{M}_{d\phi}, \boldsymbol{\mathcal{M}}_{d\phi}, \phi)) = (R\Gamma(X; \mathscr{M}_{d\phi}), R\Gamma(X; \boldsymbol{\mathcal{M}}_{d\phi}, \phi)) \qquad \text{and}$$
$$R\Gamma_c(X; (\mathscr{M}_{d\phi}, \boldsymbol{\mathcal{M}}_{d\phi}, \phi)) = (R\Gamma_c(X; \mathscr{M}_{d\phi}), R\Gamma_c(X; \boldsymbol{\mathcal{M}}_{d\phi}, \phi)).$$

Proposition 2.18. *Suppose that $\phi : X \to \mathbb{A}^1$ factors as $\phi_2 \circ \phi_1$, where $\phi_1 : X \to Y$ and $\phi_2 : Y \to \mathbb{A}^1$. Then,*

$$R\Gamma_c(X; (\mathscr{M}_{d\phi}, \boldsymbol{\mathcal{M}}_{d\phi})) \cong R\Gamma_c(Y; ((\phi_1)_!\mathscr{M}_{d\phi}, (\phi_1)_!\boldsymbol{\mathcal{M}}_{d\phi})).$$

Proof. The isomorphism $R\Gamma_c(X; \mathscr{M}_{d\phi}) \cong R\Gamma_c(Y; (\phi_1)_!\mathscr{M}_{d\phi})$ follows from the projection formula in proposition 2.6.

By lemma 2.13,

$$\begin{aligned} R\Gamma_c(Y; (\phi_1)_!\boldsymbol{\mathcal{M}}_{d\phi_2}, \phi_2) &\cong R\Gamma_c(\mathbb{A}^1; (\phi_2)_!(\phi_1)_!\boldsymbol{\mathcal{M}}_{dt}, t) \\ &\cong R\Gamma_c(\mathbb{A}^1; \phi_!\boldsymbol{\mathcal{M}}_{dt}, t). \end{aligned} \tag{2.6.3}$$

$\square$

2.7. **Künneth Formula.** In this section, we will show that there is a Künneth formula for elementary $\mathcal{D}$-modules with Betti structure. Let X and Y be smooth algebraic varieties over k. Suppose that $(\mathscr{M}, \boldsymbol{\mathcal{M}}, \phi)$ and $(\mathscr{N}, \boldsymbol{\mathcal{N}}, \psi)$ are in $\mathrm{MB}_{\mathrm{el}}(\mathcal{D}_X, M)$ and $\mathrm{MB}_{\mathrm{el}}(\mathcal{D}_Y, M)$, respectively. Define $(\mathscr{M}, \boldsymbol{\mathcal{M}}, \phi) \boxtimes (\mathscr{M}, \boldsymbol{\mathcal{M}}, \psi)$ to be the pair

$$(\mathscr{M} \boxtimes \mathscr{N}, \boldsymbol{\mathcal{M}} \boxtimes \boldsymbol{\mathcal{N}}, \phi + \psi) \in \mathrm{MB}_{\mathrm{el}}(\mathcal{D}_{X \times Y}, M).$$

Theorem 2.19. *There is a natural isomorphism*

$$R\Gamma_c(U \times V; \boldsymbol{\mathcal{M}} \boxtimes \boldsymbol{\mathcal{N}}, (\phi + \psi)) \cong R\Gamma_c(U; \boldsymbol{\mathcal{M}}, \phi) \otimes_M R\Gamma_c(V; \boldsymbol{\mathcal{N}}, g).$$

This theorem is a variation on the Thom-Sebastiani theorem (See [22] theorem 1.2.2). We will need the following elementary topological lemma:

Lemma 2.20. *Suppose that $C \cong D \times I$ is a cylinder, with I a closed interval in $\mathbb{R}$, and $\boldsymbol{\mathcal{F}}$ is a complex of sheaves that is constant on the fibers of the projection $\pi : C \to D$. Let $i : D \times \{p\} \to C$, where p is in the boundary of I, and $j : D \times I \backslash p \to C$. Then,*

$$R\Gamma(C; j_! j^* \boldsymbol{\mathcal{F}}) \cong \{0\}.$$

Proof. By assumption, there is a sheaf $\boldsymbol{\mathcal{F}}'$ on D such that $\boldsymbol{\mathcal{F}} \cong \pi^* \boldsymbol{\mathcal{F}}'$, and $\pi_* \boldsymbol{\mathcal{F}} \cong \boldsymbol{\mathcal{F}}'$. Therefore,

$$R\Gamma(C; \boldsymbol{\mathcal{F}}) \cong R\Gamma(C; i_* i^* \boldsymbol{\mathcal{F}}).$$

Since $R\Gamma(C; j_! j^* \boldsymbol{\mathcal{F}})$ is the cone of this morphism, the statement of the lemma follows. $\square$

Proof. Define $\Re(t)$ to be the real part of t. Let $H_\rho \subset \mathbb{A}^1$ is the half-plane defined by $\Re(t) > -\rho$, with $\rho >> 0$. Then, if $\boldsymbol{\mathcal{F}}$ is a constructible sheaf,

$$\begin{aligned}
R\Gamma(\mathbb{A}^1; \boldsymbol{\mathcal{F}}, t) &\cong R\Gamma(\widetilde{\mathbb{P}}^1; \beta_! \alpha_*(\boldsymbol{\mathcal{F}}) \\
&\cong H^*(\widetilde{\mathbb{P}}^1, \widetilde{\mathbb{P}}^1 - \widetilde{\mathbb{P}}^1_I; \beta_* \alpha_*(\boldsymbol{\mathcal{F}}) \\
&\cong H^*(\mathbb{A}^1, \mathbb{A}^1 - H_\rho; \boldsymbol{\mathcal{F}}) \\
&\cong R\Gamma(\mathbb{A}^1; (\beta_\rho)_! (\beta_\rho)^* \boldsymbol{\mathcal{F}}).
\end{aligned}$$

It will be helpful to refer to the following diagram throughout:

$$\begin{tikzcd}
H_\rho \times H_\rho \arrow[r, hook] \arrow[dr, "\beta_\rho \times \beta_\rho"'] & Z_\rho \arrow[r, "\gamma_\rho"] \arrow[d, "\bar{\beta}_\rho"] & V_\rho \arrow[r, two heads, "(A')|_{V_\rho}"] \arrow[d, "\beta'_\rho"] & H_{2\rho} \arrow[d, "\beta_{2\rho}"] \\
& \mathbb{A}^1 \times \mathbb{A}^1 \arrow[r, "\gamma"] & V \arrow[r, two heads, "A'"] & \mathbb{A}^1
\end{tikzcd}$$

Let r and s be the standard parameters on $\mathbb{A}^1 \times \mathbb{A}^1$. We may reduce to the case where $U = V = \mathbb{A}^1$, observing that

$$R\Gamma_c(U \times V; (\boldsymbol{\mathcal{M}}) \boxtimes (\boldsymbol{\mathcal{N}}), \phi + \psi) \cong R\Gamma(\mathbb{A}^1 \times \mathbb{A}^1; \phi_!(\boldsymbol{\mathcal{M}}) \boxtimes \psi_!(\boldsymbol{\mathcal{N}}), r + s)$$

and

$$\begin{aligned}
(2.7.1) \quad & R\Gamma_c(U; \boldsymbol{\mathcal{M}}, \phi) \otimes R\Gamma_c(V; \boldsymbol{\mathcal{N}}, \psi) \\
& \qquad \cong R\Gamma(\mathbb{A}^1; \phi_!(\boldsymbol{\mathcal{M}}), r) \otimes R\Gamma(\mathbb{A}^1; \psi_!(\boldsymbol{\mathcal{N}}), s).
\end{aligned}$$

Thus, without loss of generality, assume $\phi = r$, $\psi = s$, and both $(\mathscr{M}, \boldsymbol{\mathcal{M}}, r)$ and $(\mathscr{N}, \boldsymbol{\mathcal{N}}, s)$ are in $\mathrm{MB}_{\mathrm{el}}(\mathcal{D}_{\mathbb{A}^1}, M)$.

There is a Künneth morphism

$$\begin{aligned}
(2.7.2) \quad & R\Gamma(\mathbb{A}^1; \boldsymbol{\mathcal{M}}, r) \otimes_M R\Gamma(\mathbb{A}^1; \boldsymbol{\mathcal{N}}, s) \to \\
& \qquad R\Gamma(\mathbb{A}^1 \times \mathbb{A}^1; (\beta_\rho \times \beta_\rho)_! (\beta_\rho \times \beta_\rho)^* (\boldsymbol{\mathcal{M}} \boxtimes \boldsymbol{\mathcal{N}})).
\end{aligned}$$

We give $\mathbb{P}^2$ homogeneous coordinates x, y, z, so that $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{P}^2 \backslash V(z)$, and let $X \to \mathbb{P}^2$ be the blow-up at $(1, -1, 0)$. We may describe X as the subvariety of $\mathbb{P}^2 \times \mathbb{P}^1$ defined by $V((x + y)u - zw)$, where w, u are the homogeneous coordinates on $\mathbb{P}^1$. In particular, $t = \frac{w}{u} = \frac{x+y}{z} = r + s$, so the projection $A : X \to \mathbb{P}^1$ restricts to the addition map $a : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ away from the hyperplane at infinity.

By lemma 2.13,

$$R\Gamma_c(\mathbb{A}^1 \times \mathbb{A}^1; \boldsymbol{\mathcal{M}} \boxtimes \boldsymbol{\mathcal{N}}, r + s) \cong R\Gamma(\mathbb{A}^1; a_!(\boldsymbol{\mathcal{M}} \boxtimes \boldsymbol{\mathcal{N}}), t).$$

Define $V = A^{-1}(\mathbb{A}^1)$ and $V_\rho = A^{-1}(H_{2\rho}) \subset X$, and let $\beta'_\rho : V_\rho \hookrightarrow V$. Furthermore, let $\gamma : \mathbb{A}^1 \times \mathbb{A}^1 \to V$ be the open inclusion, and $A' : V \to \mathbb{A}^1$. Moreover, $H_{2\rho}$ is simply defined by $\Re(x) > -2\rho$. Then, since A' is proper,

$$(\beta_{2\rho})_! \beta^*_{2\rho} A'_* \gamma_! (\boldsymbol{\mathcal{M}} \boxtimes \boldsymbol{\mathcal{N}}) \cong A'_* (\beta'_\rho)_! (\beta'_\rho)^* \gamma_! (\boldsymbol{\mathcal{M}} \boxtimes \boldsymbol{\mathcal{N}}).$$

It follows that $R\Gamma_c(\mathbb{A}^1 \times \mathbb{A}^1; \boldsymbol{\mathcal{M}} \boxtimes \boldsymbol{\mathcal{N}}, x+y) \cong R\Gamma(V; (\beta'_\rho)_!(\beta'_\rho)^* \gamma_! (\boldsymbol{\mathcal{M}} \boxtimes \boldsymbol{\mathcal{N}}))$.

Now, let $Z_\rho = V_\rho \cap (\mathbb{A}^1 \times \mathbb{A}^1)$. Therefore, Z_ρ is the set of points where $\Re(x+y) > -2\rho$. Moreover, we label the inclusions $\gamma_\rho : Z_\rho \to V_\rho$ and $\bar{\beta}_\rho : Z_\rho \to (\mathbb{A}^1 \times \mathbb{A}^1)$. There is a morphism

$$(\beta'_\rho)_!(\beta'_\rho)^* \gamma_! (\boldsymbol{\mathcal{M}} \boxtimes \boldsymbol{\mathcal{N}}) \to \gamma_* (\bar{\beta}_\rho)_! (\bar{\beta}_\rho)^* (\boldsymbol{\mathcal{M}} \boxtimes \boldsymbol{\mathcal{N}}) \tag{2.7.3}$$

obtained from the adjunction map $\mathrm{id} \to \gamma_* \gamma^*$. We will show that this is an isomorphism.

Let b be a boundary point in the closures of both V_ρ and $\mathbb{A}^1 \times \mathbb{A}^1$. We consider a neighborhood of b that is homeomorphic to a polydisc $D \times D$, with coordinates z/x and t/u (as before). Let $D^\times = D - \{0\}$ and $H = \{d \in D : \Re(d) > 0\}$. Then, $V_\rho \cap (D \times D) = D \times H$ and $(\mathbb{A}^1 \times \mathbb{A}^1) \cap (D \times D) = D^\times \times D$.

$$\begin{tikzcd} D^\times \times H \arrow[r, "\gamma_1"] \arrow[d, "\beta_1"] & D \times H \arrow[d, "\beta_2"] \\ D^\times \times D \arrow[r, "\gamma_2"] & D \times D. \end{tikzcd}$$

If $\boldsymbol{\mathcal{F}}$ is the restriction of $\boldsymbol{\mathcal{M}} \boxtimes \boldsymbol{\mathcal{N}}$ to $D^\times \times H$, then lemma 2.20 implies that

$$R\Gamma((\gamma_2)_*(\beta_1)_!(\beta_1)^* \boldsymbol{\mathcal{F}}) \cong R\Gamma((\beta_2)_!(\beta_2)^*(\gamma_1)_* \boldsymbol{\mathcal{F}}) \cong \{0\}.$$

It follows that the stalks are both isomorphic to $\{0\}$ at b. Therefore, (2.7.3) is an isomorphism, and

$$R\Gamma_c(\mathbb{A}^1 \times \mathbb{A}^1; \boldsymbol{\mathcal{M}} \boxtimes \boldsymbol{\mathcal{N}}, x+y) \cong R\Gamma(\mathbb{A}^1 \times \mathbb{A}^1; (\bar{\beta}_\rho)_!(\bar{\beta}_\rho)^* (\boldsymbol{\mathcal{M}} \boxtimes \boldsymbol{\mathcal{N}})).$$

Observe that $H_\rho \times H_\rho \subset Z_\rho$, since $\Re(x) + \Re(y) > -2\rho$ if $\Re(x), \Re(y) > -\rho$. The inclusion induces a natural morphism

$$(\beta_\rho \times \beta_\rho)_!(\beta_\rho \times \beta_\rho)^* (\boldsymbol{\mathcal{M}} \boxtimes \boldsymbol{\mathcal{N}}) \to (\bar{\beta}_\rho)_!(\bar{\beta}_\rho)^* \gamma_* (\boldsymbol{\mathcal{M}} \boxtimes \boldsymbol{\mathcal{N}})$$

using the adjunction map $(\beta_\rho \times \beta_\rho)_!(\beta_\rho \times \beta_\rho)^! \to \mathrm{id}$. We compose this map with 2.7.2 to obtain a morphism

$$R\Gamma(\mathbb{A}^1; \boldsymbol{\mathcal{M}}, t) \otimes_M R\Gamma_\Phi(\mathbb{A}^1; \boldsymbol{\mathcal{N}}, \mathrm{id}) \to R\Gamma(\mathbb{A}^1; \boldsymbol{\mathcal{M}} \boxtimes \boldsymbol{\mathcal{N}}, x+y).$$

By theorem 2.12, and proposition 2.1, this morphism must be an isomorphism. $\square$

2.8. **Rapid Decay Homology.** We are interested in calculating the matrix coefficients of the isomorphism between algebraic de Rham cohomology and moderate growth cohomology, as in corollary 2.8.1. In this section, we will recall the construction of rapid decay homology found in [4], adapting it to the case of elementary $\mathcal{D}$-modules. This is a homology theory for irregular singular connections on a curve that is canonically dual to moderate growth cohomology; moreover, there is a perfect pairing between the De Rham complex of an irregular singular connection and the rapid decay chain complex of its dual connection.

Let X be a curve, $D \subset X$ a divisor, and $U = X \backslash D$. Suppose that $(\mathscr{M}, \boldsymbol{\mathcal{M}}, \phi) \in \mathrm{MB}_{\mathrm{el}}(\mathcal{D}_U, M)$, and that $\mathscr{M}$ is isomorphic to a vector bundle E with connection ∇.

In [4], Bloch and Esnault construct a 'rapid decay' chain complex $C_*(X, D; E, \nabla)$ with coefficients in $\mathrm{DR}(E)$. Roughly, a rapid decay chain consists of a pair (μ, σ), where μ is a section of $\mathrm{DR}(E)$ and $\sigma \in X$ is a simplex which only approaches D on sectors where $e^{-\phi}$ has rapid decay. We define $C_*(X, D; \boldsymbol{\mathcal{M}}, \phi)$ to be obvious reduction of structure of $C_*(X, D; E, \nabla)$ to coefficients in $\boldsymbol{\mathcal{M}}$.

Let $H_*(X, D; \boldsymbol{\mathcal{M}}, \phi)$ be the cohomology of $C_*(X, D; \boldsymbol{\mathcal{M}}, \phi)$. Define $\boldsymbol{\mathcal{M}}^\vee$ to be the dual local system to $\boldsymbol{\mathcal{M}}$. There is a pairing

$$\begin{gathered} H_*(X, D; \boldsymbol{\mathcal{M}}^\vee, -\phi) \times H^*_{\mathrm{DR}}(V; \mathscr{M}) \to \mathbb{C}, \\ (\nu, \mu^\vee \otimes \sigma) \mapsto \int_\sigma < \nu, \mu^\vee > . \end{gathered} \tag{2.8.1}$$

Since ν is a $\mathscr{M}$-valued one form, and $\mu^\vee$ is a section of the analytic dual bundle $(\mathscr{M}^{an}_{d\phi})^\vee$, $< \nu, \mu^\vee >$ is locally an analytic one form. The integral is well defined because $\mu^\vee$ has rapid decay near D.

Theorem 2.21 ([4], Theorem 0.1)**.** *The pairing above (2.8.1) is compatible with homological and cohomological equivalence, and defines a perfect pairing of finite dimensional vector spaces*

$$(,) : H_*(X, D; \boldsymbol{\mathcal{M}}^\vee_{d\phi} \otimes_M \mathbb{C}, -\phi) \times H^*_{\mathrm{DR}}(V; \mathscr{M}_{d\phi} \otimes_k \mathbb{C}) \to \mathbb{C}.$$

Corollary 2.21.1. *The period isomorphism in 2.4.1 factors through the pairing (2.8.1):*

$$\begin{aligned} H^*(R\Gamma_c(V; \boldsymbol{\mathcal{M}}_{d\phi}, \phi)) \otimes_M \mathbb{C} &\cong \left[H^*(X, D; \boldsymbol{\mathcal{M}}^\vee_{d\phi} \otimes_M \mathbb{C}, -\phi)\right]^\vee \\ &\cong H^*_{\mathrm{DR}}(V; \mathscr{M}_{d\phi}) \otimes_k \mathbb{C}. \end{aligned} \tag{2.8.2}$$

To conclude this section, we include a few relevant calculations found in [4].

Example 2.8.1 (Gamma Function)**.** *Let k be a field with a fixed imbedding in $\mathbb{C}$, and fix an element $\alpha \in k$. Define a $\mathcal{D}$-module $\mathscr{F}$ on $V = \mathrm{Spec}(k[z, z^{-1}])$ by*

$$\mathscr{F} = \mathcal{D}_{\mathbb{A}^1} / \mathcal{D}_{\mathbb{A}^1}.(z\frac{\partial}{\partial z} - \alpha).$$

$\mathscr{F}$ is isomorphic to the trivial line bundle on V with connection

$$\nabla = d + (\alpha\frac{dz}{z})\wedge,$$

so $\mathscr{F}$ has regular singular points at 0 ∞. We will consider the periods associated to the elementary $\mathcal{D}$-module $\mathscr{F}_{-dz}$.

Now, let $M = \mathbb{Q}(e^{2\pi\sqrt{-1}\alpha})$. The horizontal sections of $\mathscr{F}^{an}_{dz}$ are locally spanned by $z^{-\alpha}e^z$, so take $\boldsymbol{\mathcal{F}}_{dz}$ to be the DR sheaf for $\mathscr{M}_{dz}$ with stalks $(H^{-1}(\boldsymbol{\mathcal{F}}_{dz})_v = Mz^{-\alpha}e^z$. The monodromy of $z^{-\alpha}e^z$ around 0 and ∞ is contained in M, so this local system is well defined. Furthermore, $(H^{-1}(\boldsymbol{\mathcal{F}}_{dz}))_v$ is spanned by $z^\alpha e^{-z}$.

By direct calculation, $H^0_{\mathrm{DR}}(\mathscr{F}_{-dz})$ is one-dimensional and spanned by $\frac{dz}{z}$. Moreover, $H^i_{\mathrm{DR}}(\mathscr{F}_{-dz})$ vanishes for i nonzero. Let σ be the 'keyhole' contour in $(\mathbb{A}^1)^{an}$ that starts at infinity, traverses the positive real line, winds around 0 counterclockwise, and follows the positive real line back to infinity. σ is a z-admissible one-simplex in $\mathbb{P}^1$, and $\partial\left((z^\alpha e^{-z}) \otimes \sigma\right) = 0$.

Finally,

$$\int_\sigma z^\alpha e^{-z}\frac{dz}{z} = (e^{2\pi\sqrt{-1}\alpha} - 1)\Gamma(\alpha),$$

where Γ is the usual gamma function. Therefore, by corollary 2.21.1, the isomorphism between $H^0_{\mathrm{DR}}(V;\mathscr{F}_{-dz})\otimes_k\mathbb{C}$ and $H^0(V;\boldsymbol{\mathcal{F}}_{-dz},z)\otimes_M\mathbb{C}$ in the bases above is simply multiplication by $\frac{1}{(e^{2\pi\sqrt{-1}\alpha}-1)\Gamma(\alpha)}$.

Example 2.8.2 (Gaussian Integral)**.** *Now, with k as above, let $\mathbb{A}^1=\operatorname{Spec}(k[z])$. Define $\mathscr{N}$ to be the elementary $\mathcal{D}_{\mathbb{A}^1}$-module $\mathcal{O}_{-\alpha z dz}$. Therefore, $\mathscr{N}$ has an irregular singular point ∞. Let $\boldsymbol{\mathcal{N}}$ to be the Betti structure for $\mathscr{N}$ with coefficients in $\mathbb{Q}$ spanned by $e^{\frac{\alpha}{2}z^2}$.*

As before, $H^*_{\mathrm{DR}}(\mathbb{A}^1;\mathscr{N})$ vanishes in all degrees but 0. However, the cohomology in degree 0 is now generated by dz. Let σ be the interval $[-\infty,\infty]$ along the real line. $H_0(\mathbb{P}^1,\infty;\boldsymbol{\mathcal{N}}^\vee)$ is generated by $\sigma_\alpha\otimes e^{-\frac{\alpha}{2}z^2}$, where σ_α is σ rotated by $\sqrt{\frac{2}{\alpha}}$.

Finally,

$$\int_{\sigma_\alpha} e^{-\frac{\alpha}{2}z^2}dz=\sqrt{\frac{2\pi}{\alpha}},$$

so the period isomorphism between $H^0(\mathbb{A}^1;\mathscr{N})$ and $H^0(\mathbb{A}^1;\boldsymbol{\mathcal{N}},z^2)$ is given by multiplication by $\sqrt{\frac{\alpha}{2\pi}}$.

3. Epsilon Factors

3.1. **Determinant Lines.**

Definition 3.1. *Let k and M be subfields of $\mathbb{C}$ with fixed imbedding. We define a Picard category $\ell(k,M)$ consisting of:*

(1) *Pairs $\ell=(\ell_k,\ell_M)$ of $\mathbb{Z}$-graded lines with coefficients in k and M respectively, and a fixed isomorphism $\ell_k\otimes_k\mathbb{C}\cong\ell_M\otimes_M\mathbb{C}$. In particular, ℓ_k and ℓ_M must be in the same degree.*
(2) *$\operatorname{Hom}(\ell,\ell')=\{\phi:\ell_k\to\ell'_k\,|\,\phi_{\mathbb{C}}(\ell_M)=\ell'_M\}$, where ϕ is a k linear map and $\phi_{\mathbb{C}}$ is corresponding map on $\ell_k\otimes_k\mathbb{C}\cong\ell_M\otimes_M\mathbb{C}$. Notice that all morphisms are invertible.*
(3) *Suppose that $k'\supset k$ and $M'\supset M$. There is a tensor functor*

$$\bigotimes:\ell(k,M)\times\ell(k',M')\to\ell(k',M')$$
$$(\ell_k,\ell_M)\times(\ell'_{k'},\ell'_{M'})\mapsto(\ell_k\otimes_k\ell'_{k'},\ell_M\otimes_M\ell'_{M'}),$$

and degree is additive under $\otimes$.
(4) *There is an identity element $\mathbf{1}_{k,M}=(k,M)$ in degree 0;*
(5) *and every object $\ell=(\ell_k,\ell_M)$ has an inverse*

$$\ell^{-1}=(\operatorname{Hom}_k(\ell_k,k),\operatorname{Hom}_M(\ell_M,M))$$

This category was introduced to the author by S. Bloch. Observe that there is a natural isomorphism

$$\ell^{-1}\otimes\ell=(\operatorname{Hom}_k(\ell_k,k)\otimes_k\ell_k,\operatorname{Hom}_M(\ell_M,M)\otimes_M\ell_M)\xrightarrow{\sim}\mathbf{1}_{k,M}.$$

It is easily shown that all lines in degree 0 are isomorphic to a line of the form

$$(\xi)=(k,\xi M),$$

for some $\xi\in\mathbb{C}$, which is non-trivial whenever ξ is not in k or M.

Let $V_{k,M}$ denote a pair of r-dimensional vector spaces (V_k,V_M) with coefficients in k and M, respectively, and an isomorphism $V_k\otimes_k\mathbb{C}=V_{\mathbb{C}}\cong V_M\otimes_M\mathbb{C}$. The

determinant line of $V_{k,M}$ is a degree r element of $\ell_{k,M}$ given by taking the rth exterior power of each vector space:

$$\det(V_k, V_M) = (\wedge^r V_k, \wedge^r V_M).$$

If there is a short exact sequence $V'_{k,M} \to V_{k,M} \to V''_{k,M}$, with the necessary compatibilities, then

$$\det(V_{k,M}) \cong \det(V'_{k,M}) \otimes \det(V''_{k,M}).$$

We will follow the convention that $\det(\{0\},\{0\}) = \mathbf{1}_{k,M}$.

There is a similar construction for complexes. Now, we let $C^*_{k,M}$ denote a pair of complexes with k and M coefficients, bounded cohomology, and a quasi-isomorphism $C^*_k \otimes_k \mathbb{C} \cong C^*_M \otimes_M \mathbb{C}$. Then,

$$\det(C^*_{k,M}) = \bigotimes_{i=a}^{b} \det(H^i(C^*)_{k,M})^{(-1)^i}.$$

The degree of $\det(C^*_{k,M})$ is the Euler characteristic of C^*. Moreover, $\det(C^*_{k,M}[1]) \cong \det(C^*_{k,M})^{-1}$. When $A^*_{k,M} \to B^*_{k,M} \to C^*_{k,M}$ is a short exact sequence of complexes, the long exact sequence in cohomology allows us to construct a natural isomorphism

$$\det(B^*_{k,M}) \cong \det(A^*_{k,M}) \otimes \det(C^*_{k,M}).$$

From a geometric perspective, $\ell(k,M) = \mathrm{MB}(\mathcal{D}_{\mathrm{Spec}(k)}, M)$; therefore, we may think of $\ell(k,M)$ as the target category of various fiber functors on $\mathrm{MB}(\mathcal{D}_X, M)$. First, suppose that U/k is a smooth quasi-projective algebraic variety, $\mathscr{L}$ is a line bundle on U with integrable connection, and $(\mathscr{L}, \boldsymbol{\mathcal{L}}) \in \mathrm{MB}(\mathcal{D}_U, M)$.

Definition 3.2. *Let $x \in U(k)$, and $\iota_x : \{x\} \to U$. Define $(\mathscr{L}, \boldsymbol{\mathcal{L}}; x) \in \ell(k,M)$ to be the pair of degree 0 lines*

$$(\mathscr{L}_{d\phi}, \boldsymbol{\mathcal{L}}_{d\phi}; x) = (\iota_x^{\Delta} \mathscr{L}_{d\phi}, \iota_x^{\Delta} \boldsymbol{\mathcal{L}}_{d\phi}) \in \ell(k,M).$$

The isomorphism between the the two lines, tensored with $\mathbb{C}$, is given by (2.6.1).

Recall the global ε factor of a holonomic $\mathcal{D}$-module on a curve, defined by (1.3.2). This is clearly an object of $\ell(k,M)$. Here, we modify the global ε-factor in the case of an elementary $\mathcal{D}$-module. Suppose that U/k is a smooth connected quasi-projective variety, and $(\mathscr{E}, \boldsymbol{\mathcal{E}}, \phi) \in \mathrm{MB}_{\mathrm{el}}(\mathcal{D}_X; M)$.

Definition 3.3. *Define the global ε factor of $(\mathscr{E}, \boldsymbol{\mathcal{E}})$ by*

$$\varepsilon(U; \mathscr{E}, \boldsymbol{\mathcal{E}}) = \det(R\Gamma(U; (\mathscr{E}, \boldsymbol{\mathcal{E}}, \phi)).$$

The compactly supported version is given by

$$\varepsilon_c(U; \mathscr{E}, \boldsymbol{\mathcal{E}}) = \det(R\Gamma_c(U; (\mathscr{E}, \boldsymbol{\mathcal{E}}, \phi)).$$

Let $j : U \to X$ be a projective completion of U satisfying the conditions of definition 2.11, and let $i : D \to X$ be the complement of U. By adjunction, there are morphisms $j_! \mathscr{E} \to j_* \mathscr{E}$ and $(\beta_X)_!(\alpha_1)_*(\alpha_2)_! \boldsymbol{\mathcal{E}} \to (\beta_X)_!(\alpha_1)_*(\alpha_2)_* \boldsymbol{\mathcal{E}}$. Furthermore, let $i' : D' \to X$ be the inclusion of the regular singular points of $\mathscr{E}$. By theorem 2.12, and by adjunction, there is an isomorphism of triangles

$$\begin{tikzcd}
R\Gamma_c(U;\mathscr{E}) \arrow[r] \arrow[d] & R\Gamma(U;\mathscr{E}) \arrow[r] \arrow[d] & R\Gamma(D; i^* j_* \mathscr{E}) \arrow[d] \\
R\Gamma_c(U;\boldsymbol{\mathcal{E}},\phi) \arrow[r] & R\Gamma(U;\boldsymbol{\mathcal{E}},\phi) \arrow[r] & R\Gamma(D;(i')^*(\alpha_2)_*\boldsymbol{\mathcal{E}})
\end{tikzcd}$$

Therefore,

$$\varepsilon(U;(\mathscr{E},\boldsymbol{\mathcal{E}})) \cong \varepsilon_c(U;(\mathscr{E},\boldsymbol{\mathcal{E}})) \otimes \varepsilon(D';((i')^*\mathscr{E},(i')^*\boldsymbol{\mathcal{E}})).$$

Furthermore, if every point of D is a pole of Φ, then $\mathscr{E}$ has no regular singular points in D and $\varepsilon(U;(\mathscr{E},\boldsymbol{\mathcal{E}})) \cong \varepsilon_c(U;(\mathscr{E},\boldsymbol{\mathcal{E}}))$.

3.2. **Epsilon Factors.** As discussed in the introduction, the ε-factors consist of pairs of graded lines (ℓ_k, ℓ_M) with coefficients in k and M, respectively, along with a fixed isomorphism $\ell_k \otimes_k \mathbb{C} \cong \ell_M \otimes_M \mathbb{C}$. This is the geometric analogue of the classical ε-factor, which consists of a single line with an action of Frobenius.

These lines are determined by purely local geometric data. Suppose that $(\mathscr{M},\boldsymbol{\mathcal{M}}) \in \mathrm{MB}(X;M)$, and $\mathscr{M}$ has singularities along a divisor $D \subset X$. Fix a point $x \in X(k)$, and let $\mathfrak{o}_x$ be the completion of $\mathcal{O}_X$ at x. Furthermore, let Δ_x be an open analytic disk containing x. The localization of $(\mathscr{M},\boldsymbol{\mathcal{M}})$ to x is given by the pair

$$(\mathscr{M}_{\mathfrak{o}_x}, \boldsymbol{\mathcal{M}}_{\Delta_x}) = (\mathscr{M}\hat{\otimes}\mathfrak{o}_x, \boldsymbol{\mathcal{M}}|_{\Delta_x}).$$

Now, suppose that $\mathfrak{o}$ is a power series ring with coefficients in k and F is the field of laurent series. In the following, $\mathscr{F}$ is a $\mathcal{D}_{\mathfrak{o}}$-module, $\widetilde{\mathscr{F}}$ is an extension of $\mathscr{F}$ to an analytic disk Δ, and $\boldsymbol{\mathcal{F}}$ is a DR sheaf for $\widetilde{\mathscr{F}}$ with coefficients in M. Here, the rank of $\mathscr{F}$ is the dimension of $\mathscr{F} \otimes_{\mathfrak{o}} F$ as an F-module.

Definition 3.4. *Let $\nu \in \Omega^1_{\mathfrak{o}/k}$. A theory of ε-factors is a rule that assigns to every pair $(\mathscr{F},\boldsymbol{\mathcal{F}})$, a pair of lines $\varepsilon(\mathscr{F},\boldsymbol{\mathcal{F}};\nu) \in \ell(k,M)$ satisfying the following properties (see* [3]*, 4.9 and* [17]*, Théorème 3.1.5.4):*

1. *Whenever there are compatible triangles $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ and $\boldsymbol{\mathcal{F}}' \to \boldsymbol{\mathcal{F}} \to \boldsymbol{\mathcal{F}}''$, then*

$$\varepsilon(\mathscr{F},\boldsymbol{\mathcal{F}};\nu) \cong \varepsilon(\mathscr{F}',\boldsymbol{\mathcal{F}}';\nu) \otimes \varepsilon(\mathscr{F}'',\boldsymbol{\mathcal{F}}'';\nu).$$

   In particular, ε descends to the Grothendieck group of pairs $(\mathscr{F},\boldsymbol{\mathcal{F}})$.

2. *If $p : \mathrm{Spec}(\mathfrak{o}') \to \mathrm{Spec}(\mathfrak{o})$ is a finite map and $(\widetilde{\mathscr{F}},\widetilde{\boldsymbol{\mathcal{F}}})$ has virtual rank 0 in the Grothendieck group, then*

$$\varepsilon(p_*\widetilde{\mathscr{F}}, p_*\widetilde{\boldsymbol{\mathcal{F}}};\nu) \cong \varepsilon(\widetilde{\mathscr{F}},\widetilde{\boldsymbol{\mathcal{F}}};p^*\nu).$$

3. *if $\mathscr{F}$ is supported on the closed point of $\mathfrak{o}$, $i : \mathrm{Spec}(k) \to \mathrm{Spec}(\mathfrak{o})$, then*

$$\varepsilon(\mathscr{F},\boldsymbol{\mathcal{F}};\nu) \cong \det(i^*\mathscr{F}, i^*\boldsymbol{\mathcal{F}}).$$

4. *If $\mathscr{F}$ is nonsingular and $\mathrm{ord}(\nu) = 0$, then $\varepsilon(\mathscr{F},\boldsymbol{\mathcal{F}}) \cong \mathbf{1}_{k,M}$.*
5. *Finally, if $(\mathscr{M},\boldsymbol{\mathcal{M}})$ are defined as above and ω is a meromorphic one form on X, then*

$$\varepsilon(X;\mathscr{M},\boldsymbol{\mathcal{M}}) \cong (2\pi\sqrt{-1})^{-\mathrm{rank}(\mathscr{M})(g-1)} \otimes \left(\bigotimes_{x \in X(k)} \varepsilon(\mathscr{M}_{\mathfrak{o}_x}, \boldsymbol{\mathcal{M}}_{\Delta_x};\omega) \right).$$

The De Rham line is studied in [3], and the the Betti line is described in [2]. An unpublished result of Bloch and Esnault gives a canonical isomorphism between the De Rham and Betti lines using a local Fourier transform. In section 6, we will show that the isomorphism in rank one may be calculated by a Gauss sum.

4. Character Sheaves on $F^\times$

4.1. **Invariant $\mathcal{D}$-modules.** Suppose that G/k is an algebraic group with multiplication $\mu : G \times G \to G$, and X/k is an algebraic variety with an action of G:

$$\rho : G \times X \to X.$$

Definition 4.1. *We say that a $\mathcal{D}_G$-module $\mathscr{L}$ is invariant if there is a natural isomorphism $\alpha : \mu^\Delta \mathscr{L} \cong \mathscr{L} \boxtimes \mathscr{L}$. When $\mathscr{F}$ is a complex of $\mathcal{D}_X$-modules, then $\mathscr{F}$ is $\mathscr{L}$-twistedly equivariant (t-equivariant) whenever there is a natural isomorphism $\beta : \rho^\Delta \mathscr{F} \cong \mathscr{L} \boxtimes \mathscr{F}$, for some invariant $\mathcal{D}_G$-module $\mathscr{L}$. Finally, $\mathscr{F}$ is G equivariant when $\mathcal{O}_G$-twistedly equivariant.*

Consider the commutative diagram

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mu \times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_G \times \rho"] & G \times X \arrow[d, "\rho"] \\ G \times X \arrow[r, "\rho"] & X \end{tikzcd} .$$

In order for β to be natural, the following diagram must commute:

$$\begin{array}{ccc} (\mu \times \mathrm{id}_G)^\Delta \rho^\Delta \mathscr{F} & \xrightarrow{\gamma} & \mathscr{L} \boxtimes \mathscr{L} \boxtimes \mathscr{F} \\ \| & & \| \\ (\mathrm{id}_G \times \rho)^\Delta \rho^\Delta \mathscr{F} & \xrightarrow{\delta} & \mathscr{L} \boxtimes \mathscr{L} \boxtimes \mathscr{F}. \end{array}$$

Above, γ is the composition $[\alpha \boxtimes \mathrm{id}_{\mathscr{F}}] \circ [(\mu \times \mathrm{id}_X)]^\Delta \beta$, and δ is the composition $[\mathrm{id}_{\mathscr{L}} \boxtimes \beta] \circ [(\mathrm{id}_G \times \rho)]^\Delta \beta$. The left vertical arrow is the canonical identification

$$(\mu \times \mathrm{id}_X)^\Delta \rho^\Delta \mathscr{F} \cong (\rho \circ (\mu \times \mathrm{id}_X))^\Delta \mathscr{F} = (\rho \circ (\mathrm{id}_G \times \rho))^\Delta \mathscr{F} = (\mathrm{id}_G \times \rho)^\Delta \rho^\Delta \mathscr{F}.$$

If we replace X with G and ρ with μ, we obtain the naturality condition for α.

The following lemma is the analogue of [18], 1.9.3.

Lemma 4.2. *Let H be a connected algebraic group. Suppose that H acts freely on a variety on X and trivially on Y, and $\phi : X \to Y$ is an H-equivariant morphism. Furthermore, assume that if $y \in Y$, then there is an open neighborhood U of y such that $U \cong H \times \phi(U)$ and ϕ is the second projection. Let $\mathscr{K}$ be a holonomic $\mathcal{D}$-module on X. The following are equivalent:*

1. *$\mathscr{K}$ is trivially H-equivariant*
2. *$\mathscr{K} \cong \phi^\Delta(\mathscr{K}_1)$ for some holonomic $\mathcal{D}_Y$-module $\mathscr{K}_1$.*

Proof. Let n be the dimension of H. There is an adjunction map $\phi^* \phi_* \mathscr{M} \to \mathscr{M}$. As in (4.1.2), $\phi^* \phi_* \mathscr{M} \cong \phi^\Delta \Omega_{Y/X}(\mathscr{M})$; therefore, the cohomology of $\phi^* \phi_* \mathscr{M}$ vanishes in negative degrees. Apply the truncation functor $\tau^{\leq 0}$ to the adjunction map. Since ϕ is flat, we get a morphism $\phi^* \left(H^{-n} \left[\phi_* \mathscr{M}\right]\right) \to \mathscr{M}$. In order to show that this morphism is an isomorphism, it suffices to work locally on X.

Assume that $X = H \times Y$ and $X \xrightarrow{\phi} Y$ is the second projection (this is sufficient once the lemma is sheafified). Let $m, \pi : H \times H \times Y \to H \times Y$ be defined by $m(h, h', y) = (hh', y)$ and $\pi(h, h', y) = (h', y)$ and let $i : H \times Y \to H \times Y \times Y$ be the inclusion $i(h, y) = (h, e, y)$. Here, e is the identity of H.

Trivial equivariance implies that $\pi^!\mathcal{K} \cong m^!\mathcal{K}$. Therefore, $i^!\pi^!\mathcal{K} \cong i^!m^!\mathcal{K}$. Define $j : Y \to H \times Y$ by $j(y) = (e, y)$. Since $m \circ i = \mathrm{id}$ and $\pi \circ i(h, y) = (e, y)$, we see that $\mathcal{K} \cong \phi^! j^! \mathcal{K}$. Here is the diagram:

$$\begin{tikzcd} H \times Y \arrow[r, "i"] \arrow[d, "\phi"] & H \times H \times Y \arrow[r, "m"] \arrow[d, "\pi"] & H \times Y \\ Y \arrow[r, "j"] & H \times Y & \end{tikzcd}$$

□

Definition 4.3. *Let G be an affine algebraic group, and let $\mathfrak{g}$ be its lie algebra. Let λ be a linear functional on $\mathfrak{g}$. Suppose that λ vanishes on $[\mathfrak{g}, \mathfrak{g}]$. Define a $\mathcal{D}_{U_f}$-module*

$$\mathscr{L}_\lambda = \mathcal{D}_G / \mathscr{I}_\lambda,$$

where $\mathscr{I}_\lambda$ is the ideal generated by $\{X - \lambda(X); X \in \mathfrak{g}\}$.

Proposition 4.4. *$\mathscr{L}_\lambda$ is a line bundle. Moreover, $\mathscr{L}_\lambda$ is an invariant $\mathcal{D}_G$-module.*

Proof. Give $\mathcal{D}_G$ (and thus, $\mathscr{I}_\lambda$) the standard degree filtration. $\mathrm{gr}^1(\mathscr{I}_\lambda)$ is spanned by the image of $\mathfrak{g}$, which also generates $\mathrm{gr}^1(\mathcal{D}_G)$. It follows that

$$\mathrm{gr}(\mathscr{L}_\lambda) \cong \mathrm{gr}(\mathcal{D}_G)/\mathrm{gr}(\mathscr{I}_\lambda) \cong \mathrm{gr}^0(\mathcal{D}_G).$$

In particular, $\mathscr{L}_\lambda$ is a coherent $\mathcal{O}_G$-module of rank 1.

Now we prove invariance. $\mu^\Delta \mathscr{L}_\lambda \cong \mathcal{O}_{G\times G} \otimes_{\mathcal{O}_G} \mathscr{L}_\lambda$ as an $\mathcal{O}_{G\times G}$-module. Let v_λ be the image of 1 in $\mathcal{D}_G/\mathscr{I}_\lambda$. The isomorphism $\mathscr{L}_\lambda \boxtimes \mathscr{L}_\lambda \cong \mu^\Delta \mathscr{L}_\lambda$ is given by $v_\lambda \otimes v_\lambda \mapsto 1 \otimes v_\lambda$. To check that this is a $\mathcal{D}_{G\times G}$-module homomorphism, let $(X, Y) \in \mathfrak{g} \times \mathfrak{g}$. Recall the Baker-Campbell-Hausdorff formula states that $\mu_*(X, Y) = X + Y + \frac{1}{2}[X, Y] + \ldots$, where all of the higher order terms involve commutators. Then,

$$\begin{aligned} (X, Y)(1 \otimes v_\lambda) &= (1 \otimes \mu_*(X, Y) v_\lambda) \\ &= 1 \otimes (\left[\lambda(X + Y + \frac{1}{2}[X, Y] + \ldots)\right] v_\lambda) \\ &= (X, Y)(v_\lambda \otimes v_\lambda). \end{aligned}$$

□

Another way to describe $\mathscr{L}_\lambda$ is as follows: let ω_λ be the invariant differential form on G with the property that $\omega_\lambda(X) = \lambda(X)$ for all $X \in \mathfrak{g}$. Define a connection on $\mathcal{O}_G$ by

$$\nabla(g) = d + \omega_\lambda \wedge . \tag{4.1.1}$$

Notice that $\nabla \circ \nabla = d\omega_\lambda \wedge$ as operators from $\mathcal{O}_G$ to Ω^2_G. However, if X and Y are invariant vector fields,

$$d\omega_\lambda(X, Y) = \omega_\lambda([X, Y]) = \lambda([X, Y]) = 0.$$

Therefore, $d\omega_\lambda = 0$ and ∇ is a flat connection. The isomorphism $\mathscr{L}_\lambda \cong (\mathcal{O}_G, \nabla)$ is given by mapping the image of $1 \in \mathcal{D}_G$ to $1 \in \mathcal{O}_G$.

Proposition 4.5. *Suppose that $G \cong G_1 \times G_2$. Let $\iota_1 : G_1 \to G$ and $\iota_2 : G_2 \to G$. If $\mathscr{L}$ is an invariant $\mathcal{D}_G$-module, and $\mathscr{L}_i = \iota_i^\Delta \mathscr{L}$, then $\mathscr{L} \cong \mathscr{L}_1 \boxtimes \mathscr{L}_2$.*

Proof. Notice that the action $\rho_i : G_i \times G \to G$ factors through $G_i \times G \xrightarrow{\iota_i \times \mathrm{id}} G \times G \to G$. Therefore, $\rho_i^{\Delta}(\mathscr{L}) \cong \mathscr{L}_i \boxtimes \mathscr{L}$. Therefore, if $\rho_1 \times \rho_2 : (G_1 \times G_2 \times G \to G)$, then $(\rho_1 \times \rho_2)^{\Delta}\mathscr{L} \cong (\mathscr{L}_1 \boxtimes \mathscr{L}_2) \boxtimes \mathscr{L}$. Since $\mathscr{L}$ is invariant, there is an isomorphism $\mathscr{L}_1 \boxtimes \mathscr{L}_2 \cong \mathscr{L}$. □

Proposition 4.6. *Suppose G is a finite product of groups isomorphic to $\mathbb{G}_a$ or $\mathbb{G}_m$. Then, if $\mathscr{L}_\lambda$ is a invariant sheaf as above,*

$$R\Gamma(G; \mathscr{L}_\lambda) \cong \{0\}$$

unless $\mathscr{L}_\lambda \cong \mathcal{O}_G$.

Proof. When $G \cong \mathbb{G}_a$, $R\Gamma(G; \mathscr{L}_\lambda)$ is calculated by the complex

$$k[x] \xrightarrow{d + \lambda dx \wedge} k[x].$$

This map is an isomorphism unless $\lambda = 0$ When $G = \mathbb{G}_m$, the cohomology is calculated by

$$k[x, \frac{1}{x}] \xrightarrow{d + \lambda \frac{dx}{x} \wedge} k[x, \frac{1}{x}].$$

The cohomology is isomorphic to $\{0\}$ unless $\lambda \in \mathbb{Z}$. In that case, $\mathscr{L}_\lambda \cong \mathcal{O}_{\mathbb{G}_m}$.

In the case of a product of groups $\prod G_i$, proposition 4.5 states that $\mathscr{L}_\lambda \cong \boxtimes \mathscr{L}_{\lambda_i}$. The proposition follows from the Künneth formula for $\mathcal{D}$-modules. □

Suppose that $\pi : Y \to X$ is a principal G-bundle, where G satisfies the assumptions of proposition 4.6. Let $j_x : Y_x \to Y$ be the inclusion of the fiber over $x \in X$. Now, suppose that $\mathscr{L}$ is a holonomic $\mathcal{D}_Y$-module with the property that $j_x^* \mathscr{L}$ is G twistedly-equivariant.

Corollary 4.6.1. *With X and Y as above, let Z be the Zariski closure of the locus where $j_x^* \mathscr{L}$ is trivially G- equivariant. Then, $\pi_! \mathscr{L}$ has support on Z.*

Proof. Let $\pi_x : Y_x \to x$, and $i_x : x \to X$. Applying base change,

$$i_x^* \pi_! \mathscr{L} \cong (\pi_x)_! j_x^* \mathscr{L}.$$

Proposition 4.6 implies that $i_x^* \pi_! \mathscr{L} \cong \{0\}$ unless $x \in Z$. The corollary follows from lemma 5.5. □

Definition 4.7 (Character Sheaf)**.** *A character sheaf is a pair $(\mathscr{L}, \boldsymbol{\mathcal{L}}) \in \mathrm{MB}(\mathcal{D}_G, M)$ with the following properties: $\mathscr{L}$ is an invariant $\mathcal{D}_G$-module, and $\mu^{\Delta}(\mathscr{L}, \boldsymbol{\mathcal{L}}) \cong (\mathscr{L} \boxtimes \mathscr{L}, \boldsymbol{\mathcal{L}} \boxtimes \boldsymbol{\mathcal{L}})$.*

Let G^0 be the connected component of G. If $\mathscr{L}(G^0) \cong \mathscr{L}_\lambda$ for some functional $\lambda \in \mathfrak{g}^\vee$, we say that λ is the *infinitesimal character* of $(\mathscr{L}, \boldsymbol{\mathcal{L}})$. Furthermore, we define $(\mathscr{L}^\vee, \boldsymbol{\mathcal{L}}^\vee)$ to be the dual connection of $\mathscr{L}$ along with the dual local system of $\boldsymbol{\mathcal{L}}$.

Definition 4.8. *Let $(\ell_k, \ell_M) \in \ell(k, M)$ and $(\mathscr{L}, \boldsymbol{\mathcal{L}}) \in \mathrm{MB}(\mathcal{D}_X, M)$. Define $(\ell_k, \ell_M) \otimes (\mathscr{L}, \boldsymbol{\mathcal{L}}) = (\ell_k \otimes_k \mathscr{L}, \ell_M \otimes_M \boldsymbol{\mathcal{L}})$. If $\alpha : \mathrm{DR}(\mathscr{L}) \otimes_k \mathbb{C} \cong \boldsymbol{\mathcal{L}} \otimes_M \mathbb{C}$ and $\beta : \ell_k \otimes_k \mathbb{C} \cong \ell_M \otimes_k \mathbb{C}$ are the compatibility isomorphisms, then $\beta \otimes \alpha$ is the compatibility isomorphism for $(\ell_k, \ell_M) \otimes (\mathscr{L}, \boldsymbol{\mathcal{L}})$.*

Proposition 4.9. *Suppose that $(\mathscr{L}, \boldsymbol{\mathcal{L}}) \in \mathrm{MB}(\mathcal{D}_G, M)$ as in definition 4.7. Let $g \in G$, and let $\mu_g : G \to G$ be the map induced by right multiplication by g. Then,*

$$\mu_g^{\Delta}(\mathscr{L}, \boldsymbol{\mathcal{L}}) \cong (\mathscr{L}, \boldsymbol{\mathcal{L}}, g) \otimes (\mathscr{L}, \boldsymbol{\mathcal{L}}).$$

Proof. Let i_g be the inclusion of $g \in G$. Then, $\mu_g = \mu_g \circ (i_g \times \mathrm{id}_G)$. By definition 4.7, $(i_g \times \mathrm{id})^\Delta \mu_g^\Delta(\mathscr{L}, \boldsymbol{\mathcal{L}}) \cong (i_g \times \mathrm{id})^\Delta(\mathscr{L} \boxtimes \mathscr{L}, \boldsymbol{\mathcal{L}} \boxtimes \boldsymbol{\mathcal{L}})$, which in turn is isomorphic to $(i_g^\Delta(\mathscr{L}) \boxtimes \mathscr{L}, i_g^\Delta(\boldsymbol{\mathcal{L}}) \boxtimes \boldsymbol{\mathcal{L}})$. Finally, $(i_g^\Delta(\mathscr{L}) \boxtimes \mathscr{L}, i_g^\Delta(\boldsymbol{\mathcal{L}}) \boxtimes \boldsymbol{\mathcal{L}}) \cong (\mathscr{L}, \boldsymbol{\mathcal{L}}, g) \otimes (\mathscr{L}, \boldsymbol{\mathcal{L}})$. □

Proposition 4.10. *If $\sigma : G \to G$ is the inverse map, then*

$$\sigma^*(\mathscr{L}, \boldsymbol{\mathcal{L}}) \cong (\mathscr{L}^\vee, \boldsymbol{\mathcal{L}}^\vee).$$

Proof. Consider the maps

$$G \xrightarrow{\Delta_G} G \times G \xrightarrow{\sigma \times \mathrm{id}_g} G \times G \xrightarrow{\mu} G.$$

Notice that the image of G under composition is the identity. Then,

$$[\sigma^*(\mathscr{L}, \boldsymbol{\mathcal{L}})] \otimes_G^L (\mathscr{L}, \boldsymbol{\mathcal{L}}) \cong \Delta_G^\Delta(\sigma \times \mathrm{id}_g)^\Delta \mu^\Delta \mathscr{L}.$$

However, this implies that $[\sigma^*(\mathscr{L}, \boldsymbol{\mathcal{L}})] \otimes_G (\mathscr{L}, \boldsymbol{\mathcal{L}}) \cong (\mathcal{O}_G, M_G)$. □

If $\rho : G \times X \to X$ is a smooth G action, we say that $(\mathscr{M}, \boldsymbol{\mathcal{M}}) \in \mathrm{MB}(\mathcal{D}_X, M)$ is G-equivariant if there is a natural isomorphism $\rho^\Delta(\mathscr{M}, \boldsymbol{\mathcal{M}}) \cong (\mathcal{O}_G, M_G) \boxtimes (\mathscr{M}, \boldsymbol{\mathcal{M}})$. We consider the case where H is the additive group of an n-dimensional k-vector space, and $\phi : Y \to X$ is a principle H bundle. Suppose that $(\mathscr{M}, \boldsymbol{\mathcal{M}})$ is H-equivariant.

Proposition 4.11. *There exists a unique $(\mathscr{N}, \boldsymbol{\mathcal{N}}) \in \mathbf{D}^{\mathrm{b}}(\mathrm{MB})(\mathcal{D}_X, M)$ with the property $\phi^\Delta(\mathscr{N}, \boldsymbol{\mathcal{N}}) \cong (\mathscr{M}, \boldsymbol{\mathcal{M}})$. Furthermore,*

$$\phi_*(\mathscr{M}, \boldsymbol{\mathcal{M}}) := (\phi_*\mathscr{M}, \phi_*\boldsymbol{\mathcal{M}}) \cong (\mathscr{N}, \boldsymbol{\mathcal{N}})[n] \qquad \textit{and}$$
$$\phi_!(\mathscr{M}, \boldsymbol{\mathcal{M}}) := (\phi_!\mathscr{M}, \phi_!\boldsymbol{\mathcal{M}}) \cong (\mathscr{N}, \boldsymbol{\mathcal{N}})[-n](-n).$$

Proof. The first statement follows from lemma 4.2. There is an adjunction map $\mathcal{N} \to \phi_*\phi^*\mathcal{N}$. A lemma in [1], lecture 2.6, states that

$$\phi_*\phi^*\mathcal{N} \cong \Omega_{Y/X}(\phi^*\mathcal{M}). \tag{4.1.2}$$

For a sufficiently small neighborhood $U \subset X$, $\Omega_{Y/X}|_U \cong k$. Therefore, the adjunction map is locally an isomorphism. The same argument works using the adjunction map for $\phi_!$.

There are compatible isomorphisms $\phi_!\phi^!\boldsymbol{\mathcal{N}} \to \boldsymbol{\mathcal{N}}$ and $\boldsymbol{\mathcal{N}} \to \phi_*\phi^*\boldsymbol{\mathcal{N}}$. Since $\phi^!(\mathscr{N}, \boldsymbol{\mathcal{N}}) \cong \phi^\Delta(\mathscr{N}, \boldsymbol{\mathcal{N}})[n](n)$ by proposition 2.17,

$$\phi_!(\mathscr{M}, \boldsymbol{\mathcal{M}}) = (\phi_!\mathscr{M}, \phi_!\boldsymbol{\mathcal{M}}) \cong (\mathscr{N}, \boldsymbol{\mathcal{N}})[-n](-n).$$

□

4.2. **Character Sheaves on $F^\times$.** Recall the definitions of $\mathfrak{o}$ and $F^\times$ in section 1.4. In this section, we will consider character sheaves on subquotients of the group $F^\times$. First, we give $\mathfrak{o}$, U and $F^\times$ the structures of group schemes. Identify a formal power series $u \in \mathfrak{o}$ with

$$u = x_0 + x_1 T + \ldots + x_n T^n + \ldots,$$

so $\mathfrak{o} = \mathrm{Spec}[x_0, x_1, x_2, \ldots x_n, \ldots]$. We define $\mathfrak{p}$ to be the prime ideal of $\mathfrak{o}$ generated by T, and U to be the multiplicative group of units in $\mathfrak{o}$. There is a system of congruence subgroups $U^i \subset U$, where $U = U^0$ and $U^i = 1 + \mathfrak{p}^n$ Therefore, $U^i \subset U^{i-1}$, and

$$U = \varprojlim_i U/U^i.$$

We will use U_i to denote U/U^i.

F is the defined by the injective limit $F = \varinjlim_{i \geq 0} T^{-i}\mathfrak{o}$, and $F^\times = \coprod_{i \in \mathbb{Z}} T^i U$. In particular, there is an exact sequence

$$\{1\} \to U \to F^\times \to \mathbb{Z} \xrightarrow{\deg} \{0\}. \tag{4.2.1}$$

The degree n component of $F^\times$ is a naturally a U-torsor. For the most part, we will work with quotients $\left(T^i U\right)/U^j$ which are of finite type over k.

The lie algebra of U_j is $\mathfrak{o}/\mathfrak{p}^j$, and we denote by X_ξ the invariant vector field associated to $\xi \in \mathfrak{o}/\mathfrak{p}^j$. We fix a basis $\{X_\ell = X_{T^\ell}\}$, where

$$X_\ell(x_0, x_1, x_2, \dots, x_j) = \sum_{i=0}^{j-\ell} x_i \frac{\partial}{\partial x_{i+\ell}}. \tag{4.2.2}$$

If m_u is the map corresponding to multiplication by $u = \sum_{i=0}^{j} u_i T^i \in U$, then

$$(X_\ell m_u^* x_i)(e) = \frac{\partial}{\partial x_\ell}(\sum_{j=0}^{i} u_{i-j} x_j) = \left\{ \begin{array}{cc} u_{i-\ell} & \ell \leq i \\ 0 & \ell > i \end{array} \right. .$$

By (4.2.2), this is the same as $(X_\ell x_i)(u)$.

Let λ be a linear functional on $\mathfrak{o}$ that vanishes on $\mathfrak{p}^N$ for sufficiently large N. Define the conductor f of λ to be the smallest non-negative integer such that $\lambda(\mathfrak{p}^{f+1}) = 0$. For $M \subset \mathbb{C}$ sufficiently large, there is a unique character sheaf $(\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda) \in \mathrm{MB}(\mathcal{D}_{U_{f+1}}, M)$ associated to λ.

Proposition 4.12. *Set $\lambda_0 = \lambda(1)$, and let $M = \mathbb{Q}[e^{2\pi\sqrt{-1}\lambda_0}]$. There is a unique t-equivariant Betti structure $\boldsymbol{\mathcal{L}}_\lambda$ with coefficients in M corresponding to $\mathscr{L}_\lambda$ on U_{f+1}. Moreover, $(\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda; 1) \cong \mathbf{1}_{k,M}$.*

Proof. First, we show that $\boldsymbol{\mathcal{L}}_\lambda$ exists. The fundamental group of U_{f+1} is generated by a loop γ around $x_0 = 0$. Let $i_{\mathbb{G}_m} : \mathbb{G}_m \to U_{f+1}$ be the subgroup of constant polynomials in U_{f+1}. Up to a constant, there is one invariant vector field $X = x_0 \frac{\partial}{\partial x_0}$ on $\mathbb{G}_m$, and

$$i_{\mathbb{G}_m}^{\Delta} \mathscr{L}_\lambda \cong \mathcal{D}_{\mathbb{G}_m} / < X - \lambda_0 > .$$

The horizontal sections of $i_{\mathbb{G}_m}^{\Delta} \mathscr{L}_\lambda$ are spanned by $x_0^{\lambda_0} v_\lambda$. Therefore, $\pi_1(U_{f+1})$ acts on sections $\ell \in \left(\mathcal{H}^{-f-1}\mathrm{DR}(\mathscr{L}_\lambda)\right)_1$ by $\gamma(\ell) = e^{2\pi\sqrt{-1}\lambda_0}\ell$. Since $e^{2\pi\sqrt{-1}\lambda_0} \in M$, it is possible to construct $\boldsymbol{\mathcal{L}}_\lambda$.

By proposition 2.15 it suffices to consider $(\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda; 1)$, where 1 is the identity of U_{f+1}. Observe that $(\mu^*\mathscr{L}_\lambda, \mu^*\boldsymbol{\mathcal{L}}_\lambda; (1,1)) \cong (\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda; 1)$. Therefore, the condition in definition 4.7 implies that there is a natural isomorphism

$$(\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda; 1) \cong (\mathscr{L}_\lambda \boxtimes \mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda \boxtimes \boldsymbol{\mathcal{L}}_\lambda; (1,1)) \cong (\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda; 1)^{\otimes^2}.$$

If we pull $(\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda)$ back to $G \times G \times G$ by multiplication, there is a similar isomorphism $(\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda; 1) \cong (\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda; 1)^{\otimes^3}$. Therefore,

$$\begin{aligned} (\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda; 1) &\cong (\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda; 1)^{\otimes^3} \otimes \left[(\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda; 1)^{\otimes^2}\right]^{-1} \\ &\cong (\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda; 1) \otimes (\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda; 1)^{-1} \cong \mathbf{1}_{k,M}. \end{aligned} \tag{4.2.3}$$

□

Now, suppose that $\mathscr{L}_\lambda$ is an invariant $\mathcal{D}_{U_{f+1}}$-module. Consider $i_1 : U^1_{f+1} \subset U_{f+1}$. U^1_{f+1} is nilpotent, so there is an algebraic map $\log : U^1_{f+1} \to \mathfrak{p}/\mathfrak{p}^{f+1}$. Therefore, since $\iota^\Delta \mathscr{L}_\lambda$ is invariant,

$$i_1^\Delta \mathscr{L}_\lambda \cong \mathcal{O}_{d(\lambda \circ \log)}.$$

Furthermore, there is an isomorphism of groups $\mathbb{G}_m \times U^1_{f+1} \to U_{f+1}$ given by scalar multiplication. Define $i_{\mathbb{G}_m}$ as above. By proposition 4.5,

$$\mathscr{L}_\lambda \cong i^\Delta_{\mathbb{G}_m}(\mathscr{L}_\lambda) \boxtimes i_1^\Delta(\mathscr{L}_\lambda).$$

Let $\beta_\lambda : U_{f+1} \to \mathbb{A}^1$ be the composition of $\lambda \circ \log$ with the projection $\mathbb{G}_m \times U^1_{f+1} \to U^1_{f+1}$. We have proved the following:

Proposition 4.13. $(\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda, \beta_\lambda) \in \mathrm{MB}_{\mathrm{el}}(\mathcal{D}_{U_{f+1}}, M)$.

We will need to work with character sheaves on $F^\times_{f+1} = F^\times / U^{f+1}$. As before, there is a map $\deg : F^\times_{f+1} \to \mathbb{Z}$ with kernel U_{f+1}, so we write $F^{(n)}_{f+1}$ for the degree n component of $F^\times_{f+1}$.

A character sheaf $(\mathscr{L}, \boldsymbol{\mathcal{L}})$ on $F^\times_{f+1}$ is given by a collection of pairs $(\mathscr{L}^{(n)}, \boldsymbol{\mathcal{L}}^{(n)}) \in \mathrm{MB}(\mathcal{D}_{F^{(n)}_{f+1}}, M)$ that are invariant under multiplication $\mu : F^{(n)} \times F^{(m)} \to F^{(n+m)}$. Suppose that the infinitesimal character of $(\mathscr{L}, \boldsymbol{\mathcal{L}})$ is λ, so $(\mathscr{L}^{(0)}, \boldsymbol{\mathcal{L}}^{(0)}) \cong (\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda)$. Since $F^\times_{f+1}$ is disconnected, $(\mathscr{L}, \boldsymbol{\mathcal{L}})$ is not uniquely determined by its infinitesimal character.

Proposition 4.14. *Fix a uniformizer $T \in F^{(1)}_{f+1}$, and let μ_T correspond to multiplication by T. Then, $(\mathscr{L}, \boldsymbol{\mathcal{L}})$ is uniquely determined up to its infinitesimal character and the fiber $(\mathscr{L}, \boldsymbol{\mathcal{L}}; T) = \ell$. Furthermore, on each connected component, $(\mathscr{L}^{(n)}, \boldsymbol{\mathcal{L}}^{(n)}, \beta^{(n)}) \in \mathrm{MB}_{\mathrm{el}}(\mathcal{D}_{F^{(n)}_{f+1}}, M)$, where $\mu^*_{T^n}\beta_\lambda = \beta^{(n)}$.*

Proof. By invariance, $(\mathscr{L}^{(n)}, \boldsymbol{\mathcal{L}}^{(n)}; T^n) = \ell^{\otimes n}$ and $\mu^*_{T^n}(\mathscr{L}^{(n)}) \cong \mathscr{L}_\lambda$. Therefore, $\mathscr{L}^{(n)}$ is elementary. Proposition 2.15 implies uniqueness. □

By proposition 4.9,

$$(\mathscr{L}, \boldsymbol{\mathcal{L}}; uT) \cong (\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda; u) \otimes \ell;$$

therefore, changing T by a unit u twists ℓ by $(\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda; u)$.

We say that $(\mathscr{L}, \boldsymbol{\mathcal{L}})$ is *unramified* if the restriction of $\mathscr{L}$ to U_j is the trivial connection $\mathcal{O}_{U_j}$ for all j. Equivalently, this means that the conductor f of $\mathscr{L}$ is 0, and $\lambda(x_0 + x_1 T + \ldots) = nx_0$ for some $n \in \mathbb{Z}$. Otherwise, $(\mathscr{L}, \boldsymbol{\mathcal{L}})$ is ramified; in this case, we say that $(\mathscr{L}, \boldsymbol{\mathcal{L}})$ has tame ramification if $f = 0$ and wild ramification if $f > 0$.

Let K be the laurent series field $k((t))$, and let $\Delta^\times$ be an analytic punctured disk with a morphism $\mathrm{Spec}(K) \to \Delta^\times$. There is a canonical map $\delta : \mathrm{Spec}(K) \to F^{(1)}_{f+1}$ given (formally) by

$$\delta(t) = \frac{T}{T - t} = -\sum_{i=1}^{f} (\frac{T}{t})^i.$$

δ is canonical in the following sense: identify K with the field of Laurent series at $0 \in \mathbb{P}^1_k$, and let $G = \mathrm{Pic}(\mathbb{P}^1, a(0) + (\infty))$ be the generalized Picard group of line bundles with order a trivialization at 0 and order 1 trivialization at ∞. Then,

$$G \cong \left[F^\times_a \times \left(k((t^{-1}))/t^{-1}k[[t^{-1}]]\right)\right] / \mathcal{O}_{\mathbb{G}_m} \cong F^\times_a.$$

Under this identification, δ is the localization of the divisor map $\mathbb{P}^1 - \{0, \infty\} \to G$.

The following is a variation of theorem 3.17 in [5], and (2.26) in [6].

Proposition 4.15 (Local Class Field Theory). *Let $K = k((t))$, and let L be a K line with connection ∇_L. Let $i(L)$ be the irregularity index of L, and $a = i(L)+1$. Furthermore, let $\widetilde{L}$ be an extension of L to an analytic punctured disc, and let $\mathbf{L}$ be a Betti structure for $\widetilde{L}$ with coefficients in M. There exists a unique character sheaf $(\mathscr{L}, \boldsymbol{\mathcal{L}})$ on $F_a^\times$ with the property that $\delta^\Delta \mathscr{L} \cong L$ and $\widetilde{\delta}^\Delta(\mathscr{L}, \boldsymbol{\mathcal{L}}) \cong (\widetilde{L}, \mathbf{L})$.*

Furthermore, if we trivialize $L \cong K$, then the infinitesimal character of $(\mathscr{L}, \boldsymbol{\mathcal{L}})$, λ, is the functional with conductor $f = a-1$ defined on $\mathfrak{o}$ by

$$\lambda(g) = -\mathrm{Res}(\nabla_L(1)g).$$

Proof. Let $\lambda_i(g) = -\mathrm{Res}(t^{-i-1}g)$, $0 \leq i \leq f+1$. By explicit calculation (see [6], 2.26), $\delta^*(\omega_{\lambda_i}) = t^{-i-1}dt$. Using the description of $\mathcal{L}_\lambda$ as in line (4.1.1), it follows that $\delta^* \mathcal{L}_\lambda \cong L$. Fix a basepoint $x \in \Delta^\times$. By proposition 2.15, it suffices to choose $\boldsymbol{\mathcal{L}}_\lambda$ with the property that $(\boldsymbol{\mathcal{L}}_\lambda)_{\tilde{\delta}(x)} = (\mathbf{L})_x$. Proposition 4.14 states that this determines $\boldsymbol{\mathcal{L}}_\lambda$ uniquely. $\square$

Under class field theory, connections that admit a smooth continuation across 0 correspond to unramified character sheaves; otherwise, connections that have regular singular points correspond to the tamely ramified case and connections with irregular singular points correspond to wild ramification.

5. Gauss Sums on F

In this section, we will construct a Gauss Sum involving a character sheaf, Our motivation comes from the classical theory of Gauss sums over a local field K. In the original setting, ψ is an additive character of K, and χ is a multiplicative character with conductor f; then, the Gauss sum is defined by

$$\tau(\chi, \psi) = \sum_{u \in \mathfrak{u}/\mathfrak{u}^{f+1}} \chi(c^{-1}u)\psi(c^{-1}u),$$

for some $c \in K$.[7]

5.1. **Gauss Sums.**

First, we define an additive character of F associated to a form $\nu \in \Omega^1_{K/k}$. Define a functional $\psi_\nu : F \to \mathbb{G}_a$ by

$$\psi_\nu(g) = \mathrm{Res}(g\nu).$$

Fix two integers $N > n$, and suppose that $\mathfrak{p}^N \subset \ker(\psi_\nu)$. Then ψ_ν descends to a functional on $\mathfrak{p}^n/\mathfrak{p}^N$. Define $c(\nu)$ to be the smallest number such that $\mathfrak{p}^{-c(\nu)} \subset \ker(\psi_\nu)$; in particular $c(\nu) = \mathrm{ord}(\nu)$, the order of the zero (or pole) of ν. Let $(\mathscr{M}'_{\psi_\nu}, \boldsymbol{\mathcal{M}}'_{\psi_\nu}, \psi_\nu) \in \mathrm{MB}_{\mathrm{el}}(\mathcal{D}_{\mathfrak{p}^n/\mathfrak{p}^N}, \mathbb{Q})$ be additive invariant character sheaf defined by ψ_ν. We will denote the restriction of $(\mathscr{M}'_{\psi_\nu}, \boldsymbol{\mathcal{M}}'_{\psi_\nu}, \psi_\nu)$ to $F^{(n)}_{N-n}$ by $(\mathscr{M}_{\psi_\nu}, \boldsymbol{\mathcal{M}}_{\psi_\nu}, \psi_\nu)$.

In the following, let $g \in F^\times$, and $\mu_g : F^{(n)}_{N-n} \to F^{(n+\deg(g))}_{\deg(g)+N-n}$ be the map corresponding to multiplication by g.

Proposition 5.1. *Let $(\mathscr{M}_{\psi_\nu}, \boldsymbol{\mathcal{M}}_{\psi_\nu}) \in \mathrm{MB}_{\mathrm{el}}(\mathcal{D}_{F^{(n)}_{N-n}}, M)$ be defined as above. Then, $\mu_g^\Delta(\mathscr{M}_{\psi_\nu}, \boldsymbol{\mathcal{M}}_{\psi_\nu}) \cong (\mathscr{M}_{\psi_{g\nu}}, \boldsymbol{\mathcal{M}}_{\psi_{g\nu}})$.*

[7] In fact, the sum vanishes if c is not chosen in the proper degree, but it is otherwise independent of the choice of c.

Proof. Since $\mu_g^*(\psi_\nu) = \psi_{g\nu}$, $\mu_g^\Delta(\mathscr{M}_{\psi_\nu}) \cong \mathscr{M}_{\psi_{g\nu}}$. The same holds for $\boldsymbol{\mathcal{M}}_{\psi_\nu}$. $\square$

Definition 5.2. *Let $(\mathscr{L}, \boldsymbol{\mathcal{L}}, \beta)$ be a character sheaf on $F_{f+1}^\times$ with infinitesimal character λ, and suppose that λ has conductor f. For convenience, write $a(\mathscr{L}) = f+1$. Take $\gamma \in F^\times$ of degree $c(\nu) + a(\mathscr{L})$. Then,*

$$\tau(\mathscr{L}, \boldsymbol{\mathcal{L}}; \nu) = R\Gamma_c(\gamma^{-1}U_{f+1}; (\mathscr{L}, \boldsymbol{\mathcal{L}}, \beta) \otimes (\mathscr{M}_{\psi_\nu}, \boldsymbol{\mathcal{M}}_{\psi_\nu}, \psi_\nu)).$$

Proposition 5.3. *Let $g \in F^\times$. There is a natural isomorphism*

$$\tau((\mathscr{L}, \boldsymbol{\mathcal{L}}); g\nu) \cong \tau((\mathscr{L}, \boldsymbol{\mathcal{L}}); \nu) \otimes (\mathscr{L}, \boldsymbol{\mathcal{L}}; g)^{-1}.$$

Proof. Let $\mu_g : g^{-1}\gamma^{-1}U_{f+1} \to \gamma^{-1}U_{f+1}$ be the map corresponding to multiplication by g. Then, $\mu_g^\Delta(\mathscr{L}, \boldsymbol{\mathcal{L}}) \cong (\mathscr{L}, \boldsymbol{\mathcal{L}}) \otimes (\mathscr{L}, \boldsymbol{\mathcal{L}}, g)$ by proposition 4.9, and $\mu_g^\Delta(\mathscr{M}_{\psi_\nu}, \boldsymbol{\mathcal{M}}_{\psi_\nu}) \cong (\mathscr{M}_{\psi_{g\nu}}, \boldsymbol{\mathcal{M}}_{\psi_{g\nu}})$ by proposition 5.1. Therefore,

$$\mu_g^\Delta(\mathscr{L} \otimes \mathscr{M}_{\psi_\nu}, \boldsymbol{\mathcal{L}} \otimes \boldsymbol{\mathcal{M}}_{\psi_\nu})) \cong (\mathscr{L}, \boldsymbol{\mathcal{L}}, g) \otimes \left(\mathscr{L} \otimes \mathscr{M}_{\psi_{g\nu}}, \boldsymbol{\mathcal{L}} \otimes \boldsymbol{\mathcal{M}}_{\psi_{g\nu}}\right).$$

It follows that

$$\begin{aligned} (5.1.1) \quad & R\Gamma_c(\gamma^{-1}U_{f+1}; (\mathscr{L}, \boldsymbol{\mathcal{L}}, \beta) \otimes (\mathscr{M}_{\psi_\nu}, \boldsymbol{\mathcal{M}}_{\psi_\nu}, \psi_\nu)) \cong \\ & \qquad (\mathscr{L}, \boldsymbol{\mathcal{L}}; g) \otimes R\Gamma_c(g^{-1}\gamma^{-1}U_{f+1}; (\mathscr{L}, \boldsymbol{\mathcal{L}}, \beta) \otimes (\mathscr{M}_{\psi_{g\nu}}, \boldsymbol{\mathcal{M}}_{\psi_{g\nu}}, \psi_\nu)), \end{aligned}$$

which proves the proposition. $\square$

Fix a generator T of $\mathfrak{p}$. If the conductor of λ is f, we define the dual blob of λ to be $\delta_\lambda = \lambda(T^f)$. Notice that when $f = 0$, $\delta_\lambda = \lambda(1)$. Moreover, when $f \geq 2$, the line $((\sqrt{\delta_\lambda})^{f+1}) \in \ell(k, M)$ is independent of T: when f is odd, it is the trivial line; when f is even, a different choice of T changes δ_λ by a square.

The following is a refinement of theorem 1.1 from the introduction.

Theorem 5.4. *Suppose that $\mathscr{L}(U) \cong \mathscr{L}_\lambda$. Define $g_\lambda \in F^\times$ to be the element with the property $-\psi_{g_\lambda \nu} = \lambda$, and g_ν to be the element with the property that $\psi_\nu(g_\nu) = -1$. As before, set $a = a(\mathscr{L}) = f+1$. Then,*

$$\tau(\mathscr{L}, \boldsymbol{\mathcal{L}}; \nu) \cong \begin{cases} (e^{2\pi\sqrt{-1}\delta_\lambda} - 1)^{-1} \otimes (\Gamma(\delta_\lambda))^{-1} \otimes (\mathscr{L}, \boldsymbol{\mathcal{L}}, g_\nu) & a = 1; \\ (e^{-\operatorname{Res}(g_\lambda \nu)}) \otimes ((\sqrt{\frac{\delta_\lambda}{2\pi}})^a) \otimes (\sqrt{-1})^{\lfloor \frac{a}{2} \rfloor} \otimes (\mathscr{L}, \boldsymbol{\mathcal{L}}, g_\lambda), & a > 1. \end{cases}$$

Indeed, $(e^{2\pi\sqrt{-1}\delta_\lambda} - 1)^{-1} \in M$, so it may be omitted in the case $a = 0$.

Observe that, when $(\mathscr{L}, \boldsymbol{\mathcal{L}})$ is unramified, $\tau(\mathscr{L}, \boldsymbol{\mathcal{L}}; \nu) = (\mathscr{L}, \boldsymbol{\mathcal{L}}; g_\nu)$.

In the case $a \geq 2$, we will analyze $\tau(\mathscr{L}, \boldsymbol{\mathcal{L}}; \nu)$ using the composition of morphisms

$$U_{f+1} \xrightarrow{\pi_1} U_{\lceil \frac{f+1}{2} \rceil} \xrightarrow{\pi_2} U_{\lfloor \frac{f+1}{2} \rfloor} \xrightarrow{\pi_3} \{*\}.$$

When f is odd, $\lfloor \frac{f+1}{2} \rfloor = \lceil \frac{f+1}{2} \rceil = \frac{f+1}{2}$; in the odd case, these are the integer floor and ceiling of $\frac{f}{2}$, respectively. We write U_f^j for the image of U^j in U_f; therefore, $U_{f+1}^{\lceil \frac{f+1}{2} \rceil}$ is the kernel of π_1 and $U_{\lceil \frac{f+1}{2} \rceil}^{\lfloor \frac{f+1}{2} \rfloor}$ is the kernel of π_2.

5.2. **Vanishing Lemmas.** In order to calculate a Gauss sum, we adapt an old technique for calculating arithmetic Gauss sums. The argument uses the fact that, for any non-trivial additive character ψ, $\sum_{x\in F/\mathfrak{p}^f}\psi(x)=0$. One can show that enough terms of the Gauss sum vanish and that the remaining terms are a simple quadratic Gauss sum. We will modify this technique so it applies to the cohomology of elementary $\mathcal{D}$-modules.

Lemma 5.5. *Suppose that $\mathscr{M}^*$ is a complex of $\mathcal{D}_U$-modules with holonomic cohomology. Let ι_x be the inclusion of a point $x\in U$. Then, if $\iota_x^*\mathscr{M}\cong\{0\}$ for all $x\in U$, then there is a quasi-isomorphism $\mathscr{M}\cong\{0\}$.*

Proof. It suffices to show $H^i(\mathscr{M}^*)\cong\{0\}$. Therefore, we may assume $\mathscr{M}$ is a holonomic $\mathcal{D}_U$-module. There exists a dense open subset $j:V\to U$ with the property that $j^*\mathscr{M}$ is $\mathcal{O}_V$-coherent. Let $\iota_Y:Y\to U$ be the complement of V. Since ι_x^* coincides with the coherent sheaf pullback on V, Nakayama's lemma implies that $j^*\mathscr{M}\cong\{0\}$. Therefore, Kashiwara's theorem ([1] 1.8) implies that $\mathscr{M}\cong\iota_*\iota^!\mathscr{M}$. By induction on the dimension of U, it follows that $i_*i^!\mathscr{M}\cong\{0\}$. □

In the following, we suppose that λ has conductor $f>0$, and $(\mathscr{L}_\lambda,\boldsymbol{\mathcal{L}}_\lambda)$ is the unique character sheaf on U_{f+1} determined by λ. Furthermore, we will assume that $\psi_\nu=-\lambda$ by proposition 5.3. To simplify notation, we write $f'=\lfloor\frac{f+1}{2}\rfloor$ and $f''=\lceil\frac{f+1}{2}\rceil$.

Lemma 5.6. $(\pi_1)_!(\mathscr{L}_\lambda\otimes\mathscr{M}_{\psi_\nu})$ *has support on* $U_{f''}^{f'}$.

Proof. Let $q\in U$, and let $\iota_q:U_{f+1}^{f''}\to qU_{f+1}^{f''}$ be the inclusion of the coset generated by q. There is a group isomorphism $\mathfrak{p}^{f''}/\mathfrak{p}^{f+1}\cong U_{f+1}^{f''}$ given by

$$\mathfrak{p}^{f''}/\mathfrak{p}^{f+1}\mapsto 1+\mathfrak{p}^{f''}/\mathfrak{p}^{f+1}.$$

Multiplication by q maps $\mu_q:U_{f+1}^{f''}\to qU_{f+1}^{f''}$. Therefore, $\iota_q=\mu_q\circ\iota_1$ and

$$\begin{aligned}\iota_q^*\left(\mathscr{L}_\lambda\otimes\mathscr{M}_{\psi_\nu}\right)&\cong\iota_1^*\left(\mu_q^*(\mathscr{L}_\lambda)\otimes\mu_q^*(\mathscr{M}_{\psi_\nu})\right)\\&\cong(\mathscr{L}_\lambda,q)\otimes_k\iota_1^*\left(\mathscr{L}_\lambda\otimes\mathscr{M}_{\psi_\nu}\right).\end{aligned}$$

Identifying $U_{f+1}^{f''}$ with the additive group of $\mathfrak{p}^{f''}/\mathfrak{p}^{f+1}$, we see that $\iota_1^*\mathscr{L}_\lambda$ and $\iota_1^*\mathscr{M}_{\psi_\nu}$ are $U_{f+1}^{f''}$-invariant. In particular, by proposition 4.6,

$$R\Gamma_c\left[U_{f+1}^{f''};\iota_1^*\left(\mathscr{L}_\lambda\otimes\mu_q^*\mathscr{M}_{\psi_\nu}\right)\right]\neq\{0\}$$

only when $\mu_q^*\mathscr{M}_{\psi_\nu}=(\mathscr{L}_\lambda)^\vee$. This is only the case when

$$(\mu_q^*\psi_\nu)|_{\mathfrak{p}^{f''}/\mathfrak{p}^{f+1}}=-\lambda|_{\mathfrak{p}^{f''}/\mathfrak{p}^{f+1}}. \tag{5.2.1}$$

However, U_{f+1} has a natural action on $\mathrm{Hom}(\mathfrak{p}^{f''}/\mathfrak{p}^{f+1},k)$, and the stabilizer of ψ_ν is $U_{f+1}^{f'}$. Therefore, equation 5.2.1 is satisfied only when $q\in U_{f+1}^{f'}$.

Now, let $\bar\iota_q:qU_{f''}^{f''}\to U_{f''}$. Applying base change,

$$\bar\iota_q^*(\pi_1)_!\left[\mathscr{L}_\lambda\otimes\mathscr{M}_{\psi_\nu}\right]\cong R\Gamma_c\left[qU_{f+1}^{f''};\iota_q^*\left(\mathscr{L}_\lambda\otimes\mathscr{M}_{\psi_\nu}\right)\right].$$

By above, $\bar\iota_q^*(\pi_1)_!\left[\mathscr{L}_\lambda\otimes\mathscr{M}_{\psi_\nu}\right]$ vanishes for $q\notin U^{f'}f''$. Therefore, proposition 5.5 implies that $(\pi_1)_!\left[\mathscr{L}_\lambda\otimes\mathscr{M}_{\psi_\nu}\right]$ has support on $U_{f''}^{f'}$. □

Define a function $\varphi_\lambda : U^{f'}_{f''} \to \mathbb{G}_a$ by $\varphi(u) = \lambda(\frac{1-u^2}{2})$. When f is odd, φ is the zero function; however, when f is even, φ_λ is a non-trivial quadratic. Let $\mathcal{O}_{d\varphi_\lambda}$ be the trivial $\mathcal{D}_{U^{f'}_{f''}}$-module twisted by e^{φ_λ}, and let $\boldsymbol{\mathcal{O}}_{d\varphi_\lambda}$ be the Betti structure normalized so that $(\mathcal{O}_{d\varphi_\lambda}, \boldsymbol{\mathcal{O}}_{d\varphi_\lambda}, 1) \cong \mathbf{1}_{k,M}$.

In the following lemma, let $\iota : U^{f'}_{f+1} \to U_{f+1}$ and $\pi : U^{f'}_{f+1} \to U^{f'}_{f''}$.

Lemma 5.7. *Let $(\ell_k, \ell_M) = (\mathscr{M}_\psi, \boldsymbol{\mathcal{M}}_\psi, 1)$. Then, $\iota^*(\mathscr{L}_\lambda \otimes \mathscr{M}_\psi)$ is $U^{f''}_{f+1}$ equivariant, and*

$$\pi_!\iota^*(\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda) \otimes (\mathscr{M}_\psi, \boldsymbol{\mathcal{M}}_\psi) \cong (\ell_k, \ell_M) \otimes (2\pi\sqrt{-1})^{-f''} \otimes (\mathcal{O}_{d\varphi_\lambda}, \boldsymbol{\mathcal{O}}_{d\varphi_\lambda}).$$

Proof. First, we claim that $\iota^*\mathscr{M}_{\psi_\nu}$ is $U^{f''}_{f+1}$ twistedly equivariant. Let $\mu : U^{f''}_{f+1} \times U^{f'}_{f+1} \to U^{f'}_{f+1}$, and fix $u_1 \in \mathfrak{p}^{f''}$ and $u_2 \in \mathfrak{p}^{f'}$. Then, $u_1u_2 \in \ker(\psi_\nu)$, and

$$\mu^*\psi_\nu(1+u_1, 1+u_2) = \psi_\nu((1+u_1)(1+u_2)) = \psi_\nu(u_1) + \psi_\nu(1+u_2).$$

Therefore,

$$\mu^*\mathscr{M}_{\psi_\nu} \cong \mathscr{L}_{\psi_\nu} \boxtimes \mathscr{M}_{\psi_\nu},$$

where $\mathscr{L}_{\psi_\nu}$ is defined as in 4.3. Since $\psi_\nu = -\lambda$, it follows that $\iota^*\mathscr{L}_\lambda \otimes \mathscr{M}_{\psi_\nu}$ is *trivially* $U^{f''}_{f+1}$ equivariant.

Lemma 4.2 states that there is a unique $\mathcal{D}_{U^{f'}_{f''}}$-module $\mathscr{N}$, and Betti structure $\boldsymbol{\mathcal{N}}$, such that $\pi^\Delta\mathscr{N} \cong \iota^\Delta(\mathscr{L}_\lambda \otimes \mathscr{M}_{\psi_\nu})$ and $\pi^\Delta\boldsymbol{\mathcal{N}} \cong \iota^\Delta(\boldsymbol{\mathcal{L}}_\lambda \otimes \boldsymbol{\mathcal{M}}_{\psi_\nu})$. By proposition 4.11

$$\pi_!\iota^*(\mathscr{L}_\lambda \otimes \mathscr{M}_{\psi_\nu}, \boldsymbol{\mathcal{L}}_\lambda \otimes \boldsymbol{\mathcal{M}}_{\psi_\nu}) \cong (\mathscr{N}, \boldsymbol{\mathcal{N}})(-n) = (2\pi\sqrt{-1})^{-f''} \otimes (\mathscr{N}, \boldsymbol{\mathcal{N}}).$$

Notice that the shift from ι^* cancels the shift from $\pi_!$. We will show that

$$(\mathscr{N}, \boldsymbol{\mathcal{N}}) \cong (\ell_k, \ell_M) \otimes (\mathcal{O}_{d\varphi_\lambda}, \boldsymbol{\mathcal{O}}_{d\varphi_\lambda}).$$

Identify $U = k[[T]]^\times$, and let B be the subgroup of U/U^{f+1} consisting of

$$\{1 + xT^{f'} + yT^{2f'} : (x,y) \in \mathbb{A}^1 \times \mathbb{A}^1\}. \tag{5.2.2}$$

Multiplication is given by

$$(x_1, y_1).(x_2, y_2) = (x_1 + x_2, x_1x_2 + y_1 + y_2).$$

and there is a group homomorphism

$$\log : B \to \mathbb{G}_a \times \mathbb{G}_a$$
$$\log(x,y) = (x, y - \frac{x^2}{2}).$$

Notice that if f is odd, $B \cong \mathbb{G}_a$ and $(x,y) \equiv (x,0)$ since $yT^{2f'} \in \mathfrak{u}^{f+1}$.

Let $\iota_B : B \hookrightarrow U^{f'}_{f+1}$. If λ_B is the restriction of λ to B, then $\iota_B^*\iota^*(\mathcal{L}_\lambda) \cong \mathcal{L}_{\lambda_B}[f-2]$. Since log is an algebraic map on B, $\mathcal{L}_{\lambda_B}$ is an elementary $\mathcal{D}$-module: $\mathcal{L}_{\lambda_B} \cong \mathcal{O}_{F_\lambda}$, where

$$F_\lambda(x,y) = \lambda(\log(x,y)) = \lambda(1 + xT^{f'} + (y - \frac{x^2}{2})T^{2f'}).$$

Now, consider the quotient map $\pi_B : B \to U^{f'}/U^{f''}$. By assumption, $\psi_\nu = -\lambda$, so

$$(F_\lambda + \psi_\nu)(x,y) = \delta_\lambda(-\frac{x^2}{2}).$$

In particular, $(F_\lambda + \psi_\nu)(u) = \varphi_\lambda \circ \pi_B(u)$ for $u \in B$. Since $(\mathscr{L}_\lambda \otimes \mathscr{M}_{\psi_\nu}, \boldsymbol{\mathcal{L}}_\lambda \otimes \boldsymbol{\mathcal{M}}_{\psi_\nu}, 1) = (\ell_k, \ell_M)$, proposition 2.15 implies that

$$\iota_B^\Delta \iota^\Delta(\mathscr{L}_\lambda \otimes \mathscr{M}_{\psi_\nu}, \boldsymbol{\mathcal{L}}_\lambda \otimes \boldsymbol{\mathcal{M}}_{\psi_\nu}) \cong \pi_B^\Delta(\mathcal{O}_{d\varphi_\lambda}, \boldsymbol{\mathcal{O}}_{d\varphi_\lambda}) \otimes (\ell_k, \ell_M).$$

□

5.3. **Proof of Theorem.**

Proof. In the unramified case, $\gamma^{-1}U_0$ is a point, and ψ_ν vanishes on $\gamma^{-1}U_0$. Therefore, the de Rham line is the fiber of $\mathscr{L}$ at γ^{-1} and Betti line is the stalk of $\boldsymbol{\mathcal{L}}$.

The case $a = 1$ resembles example 2.8.1. Observe that ψ_ν gives an isomorphism between $\gamma^{-1}U_1$ and $\mathbb{G}_m$. Let g_ν be the element of $\gamma^{-1}U_1$ such that $\psi_\nu(g_\nu) = -1$. Then,

$$(\psi_\nu)_*((\mathscr{L}_\lambda \otimes \mathscr{M}_{\psi_\nu}), (\boldsymbol{\mathcal{L}}_\lambda \otimes \boldsymbol{\mathcal{M}}_{\psi_\nu})) \cong (\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda; g_\nu) \otimes (\mathscr{F}_{-dz}, \boldsymbol{\mathcal{F}}_{-dz}).$$

Above, $(\mathscr{F}_{-dz}, \boldsymbol{\mathcal{F}}_{-dz})$ is defined as in example 2.8.1. It follows from the same example that

$$\tau(\mathscr{L}, \boldsymbol{\mathcal{L}}; \nu) \cong (e^{2\pi\sqrt{-1}\delta_\lambda} - 1) \otimes (\Gamma(\delta_\lambda))^{-1} \otimes (\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda; g_\nu).$$

Now we consider the case where $a \geq 2$. By proposition 5.3,

$$\tau(\mathscr{L}, \boldsymbol{\mathcal{L}}; \nu) \cong (\mathscr{L}, \boldsymbol{\mathcal{L}}; g_\lambda) \otimes \tau(\mathscr{L}, \boldsymbol{\mathcal{L}}; g_\lambda \nu).$$

Thus, we may assume that $\psi_\nu = -\lambda$ Let π and ι be defined as in lemma 5.7. Lemma 5.6 implies that

$$\tau(\mathscr{L}, \boldsymbol{\mathcal{L}}; \nu) \cong R\Gamma_c(U_{f''}^{f'}; \pi_! \iota^* \left[(\mathscr{L}_\lambda, \boldsymbol{\mathcal{L}}_\lambda) \otimes (\mathscr{M}_{\psi_\nu}, \boldsymbol{\mathcal{M}}_{\psi_\nu})\right], \varphi_\lambda).$$

Thus, applying lemma 5.7, we deduce that

$$\tau(\mathscr{L}, \boldsymbol{\mathcal{L}}; g_\lambda \nu) \cong (\mathscr{M}_{-\lambda}, \boldsymbol{\mathcal{M}}_{-\lambda}; 1) \otimes (2\pi i)^{f'} \otimes R\Gamma_c(U_{f''}^{f'}; (\mathcal{O}_{\varphi_\lambda}, \boldsymbol{\mathcal{O}}_{\varphi_\lambda}, \phi_\lambda)).$$

When f is odd, $f'' = f'$ and $R\Gamma_c(U_{f''}^{f'}; \mathcal{O}_{\varphi_\lambda}, \boldsymbol{\mathcal{O}}_{\varphi_\lambda}) \cong 1_{k,M}$, which is the same as $(\sqrt{\delta_\lambda})^{f+1}$ since $\delta_\lambda \in k$. It suffices to calculate $R\Gamma_c(U_{f''}^{f'}; \mathcal{O}_{d\varphi_\lambda}, \boldsymbol{\mathcal{O}}_{d\varphi_\lambda}, \varphi_\lambda)$ in the case that f is even.

Recall that $\varphi_\lambda = \delta_\lambda(-\frac{x^2}{2})$. Therefore, by example 2.8.2,

$$R\Gamma_c(\mathbb{G}_a; \mathcal{O}_{d\varphi_\lambda}, \boldsymbol{\mathcal{O}}_{d\varphi_\lambda}, \varphi_\lambda) \cong (\sqrt{\frac{\delta_\lambda}{2\pi}}).$$

Moreover, $(\sqrt{\frac{\delta_\lambda}{2\pi}})^a \cong (\sqrt{\delta_\lambda}) \otimes (\sqrt{2\pi})^{-(f+1)}$ and $(\mathscr{M}_{-\lambda}, \boldsymbol{\mathcal{M}}_{-\lambda}; 1) \cong (e^{-\psi_{g_\lambda \nu}(1)})$. Note that $-\psi_{g_\lambda \nu}(1) = -\mathrm{Res}(g_\lambda \nu)$. The theorem follows by putting the pieces together. □

To end this section, we define the ε-factor of a character sheaf on $F^\times$. The definition is intended to evoke the calculation of an ε-factor of a GL_1 representation in the local field case. In the next section, we will see that these ε factors satisfy the desired global product formula.

Definition 5.8. *Let $(\mathscr{L}, \boldsymbol{\mathcal{L}})$ be a character sheaf on $F^\times$. Define*

$$\varepsilon(\mathscr{L}, \boldsymbol{\mathcal{L}}; \nu) = \tau(\mathscr{L}^\vee, \boldsymbol{\mathcal{L}}^\vee; \nu) \otimes (2\pi\sqrt{-1})^{c(\nu)}[-c(\nu) - a(\mathscr{L})] \in \ell(k, M).$$

It is worth considering the case where $(\mathscr{L}, \boldsymbol{\mathcal{L}})$ is unramified. Using local class field theory (proposition 4.15), the corresponding local Betti structure $(L, \mathbf{L})$ continues smoothly across 0 to a pair $(\bar{L}, \bar{\mathbf{L}})$. $\bar{L}$ is formally equivalent to the trivial connection on $\mathfrak{o}$, and $\mathbf{L}$ is a constant sheaf on the analytic disk Δ. There is a short exact sequence

$$(0,0) \to (j_! L, j_! \mathbf{L}) \to (\bar{L}, \bar{\mathbf{L}}) \to (i^* \bar{L}, i^* \bar{\mathbf{L}}) \to (0,0).$$

Let $\operatorname{ord}(\nu) = 0$. Then,

$$\begin{aligned} \varepsilon(j_! L, j_! \mathbf{L}; \nu) &\cong \varepsilon(\bar{L}, \mathbf{L}; \nu) \otimes \varepsilon(i^* \bar{L}, i^* \bar{L}, i^* \bar{\mathbf{L}})^{-1} \\ &\cong \varepsilon(i^* \bar{L}, i^* \bar{\mathbf{L}})^{-1} \\ &\cong (L, \mathbf{L}; 0)^{-1}. \end{aligned}$$

by definition 3.4. On the other hand, since $(\bar{L}, \bar{\mathbf{L}})$ is constant,

$$(\mathscr{L}^\vee, \boldsymbol{\mathcal{L}}^\vee; \gamma^{-1}) \cong (\mathscr{L}, \boldsymbol{\mathcal{L}}; \gamma) \cong (L, \mathbf{L}; 0)^{-1}.$$

In particular, $\varepsilon(\mathscr{L}, \boldsymbol{\mathcal{L}}; \nu) \cong \varepsilon(j_! L, j_! \mathbf{L}; \nu)$.

6. Product Formula

In this section, we will prove a product formula for the ε-factors of a rank 1 meromorphic connection on $\mathbb{P}^1$. We will use a geometric argument derived from [12] and [5] section 3. Notice that the authors cited work with curves of arbitrary genus; however, the reduction to a case resembling $\mathbb{P}^1$ is essentially proved in [5] Lemma 3.7.

6.1. **Period Determinants.** Let L be a line bundle on a smooth genus g curve X with meromorphic connection ∇. Suppose that ∇ has poles on $D = \{d_i\}$, and $V = X \backslash D$. There is a positive divisor $\mathbf{D} = \sum a_{d_i}(d_i)$ on X with the property that the complex

$$\nabla : L \to L(\mathbf{D}),$$

is quasi-isomorphic to $R\Gamma(V; j^* L)$. Notice that $a_d = i_d(L) + 1$, where i_d is the irregularity index at d.

The Euler characteristic of the cohomology may be expressed in terms of $\mathbf{D}$.

Theorem 6.1 (Index Theorem)**.**

$$\chi(R\Gamma_c(V; L)) \cong 2g - 2 + \sum_{d \in D} a_d.$$

This follows from [19], chapter 4, theorem 4.9. Now, suppose that L is a line bundle on $X = \mathbb{P}^1$ with meromorphic connection, and $\mathbf{L}$ is a Betti structure for L. We may trivialize L on V, and express $\nabla(1) = \omega + d\phi$, where ω has simple poles on D. Then, $(L, \mathbf{L}; \phi) \in \mathrm{MB}_{\mathrm{el}}(\mathcal{D}_V; M)$.

Lemma 6.2. *Let $n = -2 + \sum_{d \in D} a_d$, and let $V^{(n)}$ be the n^{th} symmetric product of V. Then, $\phi^{(n)} = \sum \phi$ defines a regular function on $V^{(n)}$, and*

$$\det(R\Gamma_c(V; (L, \mathbf{L}, \phi)) \cong (R\Gamma_c(V^{(n)}; (\mathrm{Sym}^n(L), \mathrm{Sym}^n(\mathbf{L}), \phi^{(n)})).$$

In particular, $R\Gamma_c(V^{(n)}; \mathrm{Sym}^n(L))$ is a line.

The statement for $R\Gamma_c(L)$ is proved in [5], Proposition 3.2 using the Künneth formula for integrable connections. Thus, the lemma for $(L, \mathbf{L}, \phi)$ follows by applying the Künneth formula in theorem 2.19.

6.2. **Generalized Picard group.** Suppose that $\mathbf{D} \subset \mathbb{P}^1$ is a non-reduced positive divisor. Let $\mathrm{Pic}(\mathbb{P}^1, \mathbf{D})$ be the generalized Picard group of line bundles on $\mathrm{Pic}(\mathbb{P}^1)$ with order $\mathbf{D}$ trivialization. This is the space of line bundles on $\mathbb{P}^1$ with the following equivalence relation: L_1 and L_2 are equivalent whenever there exists a rational function g such that $g \equiv 1 \pmod{\mathbf{D}}$ and $gL_1 = L_2$ ([23], Chapter 4). There is a forgetful map $\Phi : \mathrm{Pic}(\mathbb{P}^1, \mathbf{D}) \to \mathrm{Pic}(\mathbb{P}^1)$, which is precisely the degree map. Write $\mathrm{Pic}^n(\mathbb{P}^1, \mathbf{D})$ for $\Phi^{-1}(n)$.

There is an adelic (see [23], ch 5.3) description of $\mathrm{Pic}(\mathbb{P}^1, \mathbf{D})$: let I be the restricted product $\prod'_{x \in \mathbb{P}^1(k)} K_x$, where K_x is the completion of the function field of $\mathbb{P}^1$ at x. Then,

$$\mathrm{Pic}(\mathbb{P}^1, \mathbf{D}) \cong K_{\mathbb{P}^1} \backslash I / \left[\prod_{d \in \mathrm{supp}(D)} U^{a_d}(d) \times \prod'_{x \notin \mathrm{supp}(\mathbf{D})} U(x) \right]. \tag{6.2.1}$$

Above, $U(x) \subset K_x$ is the subgroup of units, and $U^a(x)$ is the degree a unit subgroup as in section 4. There is also a divisor map $\delta^{(n)} : V^{(n)} \to \mathrm{Pic}(\mathbb{P}^1, \mathbf{D})$. Specifically, $\delta^{(n)}(u_1, \ldots, u_n) = \mathcal{O}((u_1) + \ldots + (u_n))$.

There is a global analogue of proposition 4.15.

Proposition 6.3 (Global Class Field Theory)**.** *Let L be a line bundle with meromorphic connection on $\mathbb{P}^1$, and let $\mathbf{L}$ be a Betti structure with coefficients in M. Then, there exists a unique character sheaf $(\mathscr{L}, \boldsymbol{\mathcal{L}})$ on $\mathrm{Pic}(\mathbb{P}^1, \mathbf{D})$ with the property that*

$$(\delta^*(\mathscr{L}), \delta^*(\boldsymbol{\mathcal{L}})) \cong (L, \mathbf{L}).$$

Proof. The existence of $\mathscr{L}$ is proved in [5], proposition 2.17. By proposition 2.15, it suffices to choose $\boldsymbol{\mathcal{L}}$ with the property that

$$(\mathscr{L}, \boldsymbol{\mathcal{L}}; \delta(x)) \cong (L, \mathbf{L}; x)$$

for any $x \in U$. □

We return to the case where $X = \mathbb{P}^1$. Suppose that $n = \deg(\mathbf{D}) - 2 \geq 0$, and consider $\mathrm{Pic}^n(\mathbb{P}^1, \mathbf{D})$. Since $\Omega_{\mathbb{P}^1/k}(\mathbf{D})$ is rationally equivalent to $\mathcal{O}(n)$, $\mathrm{Pic}^n(\mathbb{P}^1, \mathbf{D}) = \Phi^{-1}([\Omega_{\mathbb{P}^1/k}(\mathbf{D})])$. Therefore, the set of order $\mathbf{D}$ trivializations of $\Omega_{\mathbb{P}^1/k}(\mathbf{D})$, up to isomorphism, is a $\mathrm{Pic}^{(0)}(\mathbb{P}^1, \mathbf{D})$ torsor. Furthermore, by fixing a global meromorphic form ν, we may identify this set of trivializations with $\mathrm{Pic}^n(\mathbb{P}^1, \mathbf{D})$. We let δ_ν be the composition of the divisor map with this identification.

We follow the argument in [12], Section f. Let d have multiplicity a_d in $\mathbf{D}$. Define J_d to be the $U_d^{a_d}$ torsor of differential forms with poles of order exactly a_d at d, modulo forms that are regular at d. Therefore,

$$J_d(k) = \{h_{a_d} \frac{dz}{(z-d)^{a_d}} + h_{a_d - 1} \frac{dz}{(z-d)^{a_d - 1}} + \ldots + h_1 \frac{dz}{(z-d)} : h_i \in k, h_{a_d} \neq 0\}.$$

We define $J_D = \prod_{d \in D} J_d$. There is a natural action of $\mathbb{G}_m$ on each component J_d by scalar multiplication, and we take $J'_D = J_D / \mathbb{G}_m$ to be the quotient of J_D by the diagonal action. J'_D is precisely the set of isomorphism classes of order $\mathbf{D}$ trivializations of $\Omega_{\mathbb{P}^1}(\mathbf{D})$. Thus, $J_D \cong \mathrm{Pic}^n(\mathbb{P}^1, \mathbf{D})$ for $n = \sum_{d \in \mathbf{D}} a_d - 2$.

There is a residue map $\mathrm{Res} : J_D \to \mathbb{G}_a$. Take Σ_D to be the subvariety of J_D on which Res vanishes. Since $\mathrm{Res}(\alpha\omega) = \alpha \mathrm{Res}(\omega)$ for $\alpha \in \mathbb{G}_m$, the image of Σ_D in J'_D is a codimension 1 subvariety $\Sigma'_D \subset J'_D$. Now, as above, fix a meromorphic form ν, and let (ν) be the divisor associated to the poles and zeroes of ν. Suppose that

$(v_i) \in \mathrm{Sym}^n(V)$. There is a rational function $Q^{\nu}_{(v_i)}$, unique up to a scalar, with divisor class $\sum_i^n (v_i) - \mathbf{D} - (\nu)$. The divisor map $\delta_\nu : \mathrm{Sym}^n(V) \to \Sigma'_D$ is given by

$$\delta_\nu((v_i)) = \prod_{d\in D} \overline{(Q^{\nu}_{(v_i)}\nu)},$$

where $\overline{(Q^{\nu}_{(v_i)}\nu)}$ is the image of $(Q^{\nu}_{(v_i)}\nu)$ in J'_D. By the residue theorem and dimension, δ_ν is an isomorphism between $V^{(n)}$ and Σ'_D. Therefore, $\phi^{(n)}$ defines a regular function on Σ'_D.

Finally, by proposition 6.3, there is an invariant $\mathcal{D}$-module $\mathscr{L}$ and a Betti structure $\boldsymbol{\mathcal{L}}$ on $\mathrm{Pic}^{(n)}(X;\mathbf{D})$ such that $\delta_\nu^!\mathscr{L} \cong \mathrm{Sym}^n(\mathscr{F})$ and $\delta_\nu^!\boldsymbol{\mathcal{L}}[-1] \cong \mathrm{Sym}^n(\boldsymbol{\mathcal{F}})$. We have proved the following lemma:

Lemma 6.4. *Let $\iota'_\Sigma : \Sigma'_D \to J'_D$. Then,*

$$\left[R\Gamma_c(J'_D, ((\iota_\Sigma)_*\iota_\Sigma)^*\mathscr{L}), (\iota'_\Sigma)_*(\iota'_\Sigma)^*\boldsymbol{\mathcal{L}}, \phi^{(n)})\right] \cong \det(R\Gamma_c(V; L, \mathbf{L}, \phi)).$$

6.3. **Proof of Product Formula.** We return to the character sheaf $(\mathscr{L}, \boldsymbol{\mathcal{L}})$ on $\mathrm{Pic}(\mathbb{P}^1, \mathbf{D})$. Let $F^\times_{(d)}$ be the field of Laurent series at $d \in D$, modulo the unit ideal U^{a_d}. Let $\mathbf{D}_\nu$ be the divisor $\mathbf{D} + (\nu)$, and let D_ν be the union of the support of $\mathbf{D}$ with the support of (ν).

By the description of $\mathrm{Pic}(\mathbb{P}^1, \mathbf{D})$ in (6.2.1), there is a natural surjection $\pi : \prod_{d\in\mathbf{D}_\nu} F^\times_{(d)} \to \mathrm{Pic}(\mathbb{P}^1, \mathbf{D})$.

Proposition 6.5. *Let $(\mathscr{L}_d, \boldsymbol{\mathcal{L}}_d)$ be the character sheaf on $F^\times_{(d)}$ determined by local class field theory as in proposition 4.15. Then,*

$$\pi^\Delta(\mathscr{L}, \boldsymbol{\mathcal{L}}) \cong \boxtimes_{d\in D_\nu}(\mathscr{L}_d, \boldsymbol{\mathcal{L}}_d).$$

Proof. Let $\delta : U \to \mathrm{Pic}(\mathbb{P}^1, \mathbf{D})$ be the divisor map, and suppose that T is a parameter at d. Then, after multiplying by the global function $\frac{T}{T-T(u)}$, $\delta(u) = \frac{T}{T-T(u)}$ in $F^\times_d$, and the identity in all other components. Notice that this is precisely the local divisor map from proposition 4.15.

Let $\Delta^\times_d$ be a formal disk around d. Let $\delta_d : \Delta^\times_d \to F^\times_d$ be the composition of the inclusion $\Delta^\times_d \to \mathbb{P}^1$ with the divisor map above. Therefore, since δ_d factors through π, it follows that the restriction of $\pi^\Delta(\mathscr{L}, \boldsymbol{\mathcal{L}})$ to $F^\times_d$ must be $(\mathscr{L}_d, \boldsymbol{\mathcal{L}}_d)$. The proposition follows from proposition 4.5. $\square$

Corollary 6.5.1. *Let $(\mathscr{L}^{(n)}, \boldsymbol{\mathcal{L}}^{(n)})$ be the restriction of $(\mathscr{L}, \boldsymbol{\mathcal{L}})$ to the degree n component of* $\mathrm{Pic}(\mathbb{P}^1, \mathbf{D})$. *Then, there is an invariant regular function $\beta^{(n)}$:* $\mathrm{Pic}(\mathbb{P}^1, \mathbf{D}) \to \mathbb{A}^1$ *such that $(\mathscr{L}^{(n)}, \boldsymbol{\mathcal{L}}^{(n)}, \beta) \in$* $\mathrm{MB_{el}}(\mathcal{D}_{\mathrm{Pic}(\mathbb{P}^1,\mathbf{D})}, M)$.

Proof. Let m_d be integers such that $\sum_{d\in D_\nu} m_d = n$, and let $\gamma_d \in F^\times_{(d)}$ be an element of degree m_d. There is a a principal $\mathbb{G}_m$ bundle

$$\rho : \prod_{d\in D_\nu} \gamma_d U_{a_d}(d) \to \mathrm{Pic}^{(n)}(\mathbb{P}^1, \mathbf{D}).$$

Proposition 6.5 implies that

$$\rho^\Delta(\mathscr{L}, \boldsymbol{\mathcal{L}}) \cong \boxtimes_{d\in D_\nu}(\mathscr{L}^{(m_d)}_d, \boldsymbol{\mathcal{L}}^{(m_d)}_d).$$

Recall that $(\mathscr{L}_d^{(m_d)}, \boldsymbol{\mathcal{L}}_d^{(m_d)}, \beta_d^{(m_d)}) \in \mathrm{MB}_{\mathrm{el}}(\mathcal{D}_{\gamma U_{a_d}(d)}, M)$. The sheaf in (6.3) is trivially $\mathbb{G}_m$ equivariant, so in particular $\sum \beta_d^{(m_d)}$ is $\mathbb{G}_m$ equivariant. Therefore, $\sum \beta_d^{(m_d)}$ descends to a function $\beta^{(n)}$ on $\mathrm{Pic}^{(n)}(\mathbb{P}^1, \mathbf{D})$. Take $\beta(f) = \beta^{(\deg(f))}(f)$. $\square$

For each $d \in D_\nu$, choose $\gamma_d \in F_{(d)}^\times$ of degree $a_d + c(\nu)$. be as above, and define $\alpha : \prod_{d\in\mathbf{D}} \gamma_d U_{a_d}(d) \to J_D$ by

$$\alpha(f)_d = (f^{-1}\nu)_d.$$

The diagram

$$\begin{tikzcd} \prod_{d\in\mathbf{D}} \gamma_d U_{a_d}(d) \arrow[r, "\alpha"] \arrow[d, "\pi"] & J_D \arrow[d, "\pi'"] \\ \mathrm{Pic}^n(\mathbb{P}^1;\mathbf{D}) \arrow[r, "\delta_\nu"] & J'_D \end{tikzcd}$$

commutes, the vertical arrows are surjections, and the horizontal arrows are isomorphisms. Let σ_d be the inverse map on $F_{(d)}^\times$, and take ψ to be the functional on J_D defined by summing over the residues of each component. Then, the zeroes of ψ are given by Σ_D, and $\alpha^*\psi = \sum_{d\in\mathbf{D}} \psi_\nu \circ \sigma|_{\gamma_d U_{a_d}(d)}$. By proposition 4.10,

$$(6.3.1) \quad (\pi')^\Delta(\delta_\nu)_*(\mathscr{L}, \boldsymbol{\mathcal{L}}, \beta) \cong \boxtimes_{d\in D_\nu} ((\mathscr{L}_d^\vee)^{(-c(\nu)-a(d))}, (\boldsymbol{\mathcal{L}}_d^\vee)^{(-c(\nu)-a(d))}, -\beta_d).$$

Let $\iota_\Sigma : \Sigma \to J_D$. Define an elementary $\mathcal{D}_{J_d}$-module $\mathcal{O}_{d\psi}$, and let $\boldsymbol{\mathcal{O}}_{d\psi}$ be the Betti structure with the property that $\iota_\Sigma^*(\mathcal{O}_{d\psi}, \boldsymbol{\mathcal{O}}_{d\psi}) \cong (\mathcal{O}_{\Sigma_D}, \mathbb{Q}_{\Sigma_D})[1]$.

Lemma 6.6. *There is a distinguished triangle*

$$\mathcal{O}_{\Sigma'_D} \to \pi_! \mathcal{O}_{d\psi} \to \mathcal{O}_{J'_D}.$$

Proof. Use the $\mathbb{G}_m$ action on J_D to construct $\overline{J}_D = J_D \times_{\mathbb{G}_m} \mathbb{G}_a$. Therefore, the induced map $\bar\pi : \overline{J}_D \to J'_D$ is a principle $\mathbb{G}_a$ bundle. Extend ψ by zero to a function $\bar\psi$ on $\overline{J}_D$. Let $j : J_D \to \overline{J}_D$, and let $i_Z : Z \subset \overline{J}_D$ be the inclusion of the zero section of the $\mathbb{G}_a$-bundle. There is a distinguished triangle

$$j_!\mathcal{O}_{d\psi} \to \mathcal{O}_{d\bar\psi} \to (i_Z)_*(i_Z)^*\mathcal{O}_{d\bar\psi}.$$

Since $\bar\psi$ is identically 0 on Z, $(i_Z)_*(i_Z)^*\mathcal{O}_{d\bar\psi} \cong \mathcal{O}_Z[1]$; therefore, we obtain a distinguished triangle

$$\mathcal{O}_Z \to j_!\mathcal{O}_{d\psi} \to \mathcal{O}_{d\bar\psi}.$$

The desired triangle is obtained by applying $\bar\pi_!$ to the above triangle. The induced map $Z \to J'_D$ is an isomorphism, so it suffices to show that

$$(6.3.2) \qquad \bar\pi_!\mathcal{O}_{\bar\psi} \cong \mathcal{O}_{\Sigma'_D}[1].$$

The function ψ is linear on the fibers of $\bar\pi$. Now, by corollary 4.6.1, the support of $\bar\pi_!\mathcal{O}_{\bar\psi}$ is contained in Σ'_D. Let $\overline{\Sigma}_D = \bar\pi^{-1}(\Sigma'_D)$, and $\iota_{\overline{\Sigma}_D} : \overline{\Sigma}_D \to \overline{J}_D$. Since $\iota_\Sigma^*\mathcal{O}_{\bar\psi} \cong \mathcal{O}_{\overline{\Sigma}_D}[1]$, proposition 4.11 implies that

$$\bar\pi_!\mathcal{O}_{\overline{\Sigma}_D}[1] \cong \bar\pi_!\bar\pi^!\mathcal{O}_{\Sigma'_D} \cong \mathcal{O}_{\Sigma'_D}.$$

This verifies 6.3.2. $\square$

Finally, we will need to use the following lemma:

Lemma 6.7. $R\Gamma_c(J'_D; \mathscr{L}) \cong \{0\}$.

Proof. Let $G = \prod_{d\in\mathbf{D}} U_{a_d}(d)$. Then, J_D is a G-torsor, and J'_D is a $G/\mathbb{G}_m$-torsor. Let $\mathscr{L}_G$ be the invariant $\mathcal{D}_G$-module corresponding to G. Fix an isomorphism $G \cong J_D$ by choosing a point $\nu \in J_D$. Then, if ℓ is the fiber of $\mathscr{L}$ at ν, $\mathscr{L}_G \cong \ell \otimes_k \mathscr{L}$. Similarly, define $\mathscr{L}'_G$ on $G/\mathbb{G}_m$. If $\pi_G : G \to G/\mathbb{G}_m$, $\pi_G^\Delta \mathscr{L}'_G \cong \mathscr{L}_G$. It suffices to show that $R\Gamma_c(G/\mathbb{G}_m; \mathscr{L}'_G)$ vanishes.

Let $x \in \mathbf{D}$, and

$$G' = U_x^1 \times \prod_{\substack{d\in\mathbf{D}\\ d\neq d}} U_{a_d}(d) \subset G.$$

By theorem 6.1, the dimension of G is at least 2, so G' is non-trivial. Moreover, $i_G : G' \to G$ is a section of $G \to G/\mathbb{G}_m$. Therefore, we may identify G' with $G/\mathbb{G}_m$. Let $H = U_{a_d}^{a_d-1}(d)$ be a non-trivial subgroup of G', and $\pi_H : G' \to G'/H$. Since a_d is the conductor of $\mathscr{L}_d$, the restriction of $\mathscr{L}_d$ to the fibers of π_H is non-trivial. Therefore, corollary 4.6.1 implies that $(\pi_H)_! \mathscr{L}'_G \cong \{0\}$. By composition of push-forward, $R\Gamma_c(G'; \mathscr{L}'_G) \cong 0$. □

Theorem 6.8 (Product Formula)**.** *Let $(L, \mathbf{L})$ be a holonomic $\mathcal{D}_{\mathbb{P}^1}$-module, with singular points lying in $\mathbf{D} \subset \mathbb{P}^1$. Let $V = \mathbb{P}^1 \backslash \mathbf{D}$. Furthermore, let $(\mathscr{L}_x, \boldsymbol{\mathcal{L}}_x)$ be the character sheaf defined by local class field theory and suppose that ν is a meromorphic form on $\mathbb{P}^1$. Then,*

$$\varepsilon_c(V; L, \mathbf{L}) \cong (2\pi\sqrt{-1}) \otimes \bigotimes_{x\in\mathbb{P}^1} \varepsilon(\mathscr{L}_x^\vee, \boldsymbol{\mathcal{L}}_x^\vee; \nu).$$

Proof. Recall the definition of $\varepsilon(\mathscr{L}, \boldsymbol{\mathcal{L}}; \nu)$ in definition 5.8. Let $c_x(\nu) = \mathrm{ord}_x(\nu)$, where $\mathrm{ord}_x(\nu)$ is the degree of the zero or pole of ν at x. Therefore,

$$\varepsilon_c(\mathscr{L}_x^\vee, \boldsymbol{\mathcal{L}}_x^\vee; \nu) = (2\pi\sqrt{-1})^{c_x(\nu)} \otimes \tau(\mathscr{L}_x, \boldsymbol{\mathcal{L}}_x; \nu)[-c_x(\nu) - a_x]$$

Since ν is a one form, $\sum_{x\in\mathbb{P}^1} c_x(\nu) = -2$. Moreover, if $x \in D$ $f_x = m_x$, so the degree of $\bigotimes_{x\in\mathbb{P}^1} \varepsilon(\mathscr{L}_x^\vee, \boldsymbol{\mathcal{L}}_x^\vee; \nu)$ is

$$-2 + \sum_{x\in D} (1 + m_x).$$

By theorem 6.1, this is the same as the degree of $\varepsilon_c(V; L, \mathbf{L})$. Using lemma 6.2, it suffices to show that

$$R\Gamma_c(V^{(n)}; (\mathrm{Sym}^n(L), \mathrm{Sym}^n(\boldsymbol{\mathcal{L}}), \phi^{(n)}) \cong (2\pi\sqrt{-1})^{-1} \bigotimes_{x\in X} \tau(\mathscr{L}_x^\vee, \boldsymbol{\mathcal{L}}_x^\vee; \nu).$$

The pair $(\mathcal{O}_\psi, \boldsymbol{\mathcal{O}}_\psi)$ on J_D is isomorphic to the product $(\boxtimes_{d\in\mathbf{D}_\nu} \mathcal{O}_{\psi_\nu}, \boxtimes_{d\in\mathbf{D}_\nu} \boldsymbol{\mathcal{O}}_{\psi_\nu})$. By the theorem 2.19,

$$\bigotimes_{x\in\mathbb{P}^1} \tau(\mathscr{L}_x^\vee, \boldsymbol{\mathcal{L}}_x^\vee; \nu) \cong \tag{6.3.3}$$

$$R\Gamma_c\left(\prod_{d\in D_\nu} \gamma_d U_{a_d}(d); \boxtimes_{d\in D_\nu} \left[(\mathscr{L}_d^\vee, \boldsymbol{\mathcal{L}}_d^\vee, -\beta_d) \otimes (\mathscr{M}_{\psi_\nu}, \boldsymbol{\mathcal{M}}_{\psi_\nu}, \psi_\nu)\right]\right).$$

However, by line (6.3.1),

$$(\pi')^\Delta(\mathscr{L}, \boldsymbol{\mathcal{L}}) \cong \boxtimes_{d\in D_\nu} (\mathscr{L}_d^\vee, \boldsymbol{\mathcal{L}}_d^\vee, -\beta_d),$$

and

$$\boxtimes_{d\in D_\nu} (\mathcal{O}_{\psi_\nu}, \boldsymbol{\mathcal{O}}_{\psi_\nu}, \psi_\nu) \cong (\mathscr{M}_{d\psi}, \boldsymbol{\mathcal{M}}_{d\psi}, \psi).$$

Therefore,

$$(6.3.4)\quad (2\pi\sqrt{-1})^{-1}\bigotimes_{x\in\mathbb{P}^1}\tau(\mathscr{L}_x^\vee,\boldsymbol{\mathcal{L}}_x^\vee;\nu)\cong R\Gamma_c(J_D;(\pi')^\Delta(\mathscr{L},\boldsymbol{\mathcal{L}},\beta)(1)\otimes(\mathcal{O}_{d\psi},\boldsymbol{\mathcal{O}}_{d\psi},\psi))$$

On the other hand, lemma 6.4 implies that

$$R\Gamma_c\left[J'_D;((\iota'_\Sigma)_*(\iota'_\Sigma)^*\mathscr{L}),(\iota'_\Sigma)_*(\iota'_\Sigma)^*\boldsymbol{\mathcal{L}}),\phi^{(n)})\right]\cong\det(R\Gamma_c(V;(L,\mathbf{L},\phi)).$$

Therefore, the left hand side of 6.3 is a line.

It suffices to show that

$$(6.3.5)\quad R\Gamma_c(J_D;(\pi')^\Delta(\mathscr{L},\boldsymbol{\mathcal{L}},\beta)(1)\otimes(\mathcal{O}_{d\psi},\boldsymbol{\mathcal{O}}_{d\psi},\psi))\cong R\Gamma_c\left(J'_D;\left((\iota'_\Sigma)_*(\iota'_\Sigma)^*\mathscr{L},(\iota'_\Sigma)_*(\iota'_\Sigma)^*\boldsymbol{\mathcal{L}}),\phi^{(n)}\right)\right).$$

Tensoring $\mathscr{L}$ with the triangle from lemma 6.6, we obtain a distinguished triangle

$$((\iota'_\Sigma)_*(\iota'_\Sigma)^*\mathscr{L})[-1]\to\mathscr{L}\otimes\pi_!\mathcal{O}_\psi\to\mathscr{L}.$$

By lemma 6.7,

$$R\Gamma_c(J'_D;(\iota'_\Sigma)_*(\iota'_\Sigma)^*\mathscr{L}[-1])\cong R\Gamma_c(J'_D;\mathscr{L}\otimes\pi_!\mathcal{O}_\psi).$$

Using the projection formula in proposition 2.6, $\mathscr{L}\otimes\pi'_!\mathcal{O}_\psi\cong\pi'_!((\pi')^\Delta\mathscr{L}\otimes\mathcal{O}_\psi)$. This proves (6.3) in the $\mathcal{D}$-module case.

Now, we will work with $\boldsymbol{\mathcal{L}}$. Recall, from the proof of lemma 6.6, that $\bar{\pi}:\overline{\Sigma}_D\to\Sigma'_D$ is a $\mathbb{G}_a$-bundle. Furthermore, as above, there is a natural isomorphism

$$(6.3.6)\quad R\Gamma_c(J_D;(\pi')^\Delta(\mathscr{L},\boldsymbol{\mathcal{L}},\beta)(1)\otimes(\mathcal{O}_{d\psi},\boldsymbol{\mathcal{O}}_{d\psi},\psi))\cong R\Gamma_c(\overline{\Sigma}_D;(\iota_{\overline{\Sigma}_D})^*(\mathscr{L},\boldsymbol{\mathcal{L}},\beta)(1)).$$

Finally, proposition 4.11 implies that

$$\pi'_!(\iota_{\overline{\Sigma}})_!\iota^*_{\overline{\Sigma}}(\mathscr{L},\boldsymbol{\mathcal{L}})(1)\cong(\iota_{\Sigma'})_!\iota^*_{\Sigma'}(\mathscr{L},\boldsymbol{\mathcal{L}}).$$

This confirms line (6.3.5). □

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